The Tabular Method for Repeated Integration by Parts

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1 Integration by Parts

Given two functions f, g defined on an open interval I, let $f = f^{(0)}, f^{(1)}, f^{(2)}, \dots, f^{(n)}$ denote the first n derivatives of f[1] and $g = g^{(0)}, g^{(-1)}, g^{(-2)}, \dots, g^{(-n)}$ denote n antiderivatives of g.[2] Our main result is the following generalization of the standard integration by parts rule.[3]

Theorem 1. *For $n \in \mathbb{N}$,*

$$\int f(x)g(x)\,dx = \sum_{j=0}^{n-1}(-1)^j f^{(j)}(x)\,g^{(-(j+1))}(x) + (-1)^n \int f^{(n)}(x)g^{(-n)}(x)\,dx. \tag{1}$$

Proof. We induct on n. When $n = 1$ the formula becomes

$$\int f(x)g(x)\,dx = f(x)g^{(-1)}(x) - \int f^{(1)}(x)g^{(-1)}(x)\,dx$$

which is the result of integration by parts with the choices $u = f$ and $dv = g\,dx$.

So now assume the formula (1) holds for some $n \geq 1$. In the integral we integrate by parts, taking $u = f^{(n)}$ and $dv = g^{(-n)}\,dx$. Then $du = f^{(n+1)}\,dx$ and $v = g^{(-(n+1))}$ so that

$$\int f^{(n)}(x)g^{(-n)}(x)\,dx = f^{(n)}(x)g^{(-(n+1))}(x) - \int f^{(n+1)}(x)g^{(-(n+1))}(x)\,dx.$$

Substituting this into (1) and collecting the "integrated" term into the sum we end up with

$$\sum_{j=0}^{n}(-1)^j f^{(j)}(x)\,g^{(-(j+1))}(x) + (-1)^{n+1} \int f^{(n+1)}(x)g^{(-(n+1))}(x)\,dx,$$

which is precisely (1) with $n+1$ replacing n. Hence the $n+1$ case holds if the n case does. By induction, the formula is valid for all $n \in \mathbb{N}$.

$\square$

As an immediate consequence we have the next result, which tells us how to obtain the antiderivative of fg completely in the case that f eventually differentiates to zero, i.e. when f is a polynomial.

[1] We assume f is sufficiently smooth to be differentiated arbitrarily often on I.

[2] For this we need only assume that g is continuous on I.

[3] Keep in mind that for a function f, an exponent in parentheses indicates a derivative of a certain order, with negative exponents indicating antiderivatives. We are *not* simply raising f to powers.

Corollary 1. *Suppose that* $f^{(n)} \equiv 0$. *Then*

$$\int f(x)g(x)\,dx = \sum_{j=0}^{n-1} (-1)^j f^{(j)}(x)\, g^{(-(j+1))}(x) + C.$$

The result of Theorem 1 is perhaps most easily implemented using a table. In one column we list f and its first n derivatives. In an adjacent column we list g and its first n antiderivatives. We label the columns as u and dv in keeping with the standard notation used when integrating by parts. We then multiply diagonally down and to the right to construct the summands of (1), and then alternately add and subtract them to get the correct signs. At the final level, we multiply directly across, continue the alternation of signs, but integrate the resulting term to get the integral appearing in (1). See the diagram below.

$$\begin{tikzcd}
u & dv \\
f \arrow[dr, "+"] & g \\
f^{(1)} \arrow[dr, "-"] & g^{(-1)} \\
f^{(2)} \arrow[dr, "+"] & g^{(-2)} \\
f^{(3)} \arrow[dr, "-"] & g^{(-3)} \\
\vdots & \vdots \\
f^{(n-1)} \arrow[dr, "(-1)^{n-1}"] & g^{(-(n-1))} \\
f^{(n)} \arrow[r, dashed, "(-1)^n"'] & g^{(-n)}
\end{tikzcd}$$

Each solid arrow indicates a multiplication between the terms at either end, the result of which is then added or subtracted to the to the antiderivative according to the sign shown with the arrow. The final dashed arrow indicates that the two terms at either end are to be multiplied together, *integrated*, and then added or subtracted from the antiderivative according to the sign of $(-1)^n$. This gives us

$$fg^{(-1)} - f^{(1)}g^{(-2)} + f^{(2)}g^{(-3)} - f^{(3)}g^{(-4)} + \cdots + (-1)^{n-1}f^{(n-1)}g^{(-n)} + (-1)^n \int f^{(n)}g^{(-n)}\,dx,$$

in agreement with (1).

Example 1. Antidifferentiate $(x^3 + 2x - 1)\cos(4x)$.

Solution. We take $f(x) = x^3 + 2x - 1$, $g(x) = \cos(4x)$ and construct the table above:

$$
\begin{tikzcd}
u & dv \\
x^3+2x-1 \arrow[dr, "+"] & \cos(4x) \\
3x^2+2 \arrow[dr, "-"] & \frac{1}{4}\sin(4x) \\
6x \arrow[dr, "+"] & \frac{-1}{16}\cos(4x) \\
6 \arrow[dr, "-"] & \frac{-1}{64}\sin(4x) \\
0 \arrow[r, dashed, "+"'] & \frac{1}{256}\cos(4x)
\end{tikzcd}
$$

The antiderivative is therefore

$$\frac{1}{4}(x^3+2x-1)\sin(4x)+\frac{1}{16}(3x^2+2)\cos(4x)-\frac{3x}{32}\sin(4x)-\frac{3}{128}\cos(4x)+\int 0\,dx =$$

$$\boxed{\left(\frac{1}{4}x^3+\frac{13}{32}x-\frac{1}{4}\right)\sin(4x)+\left(\frac{3}{16}x^2+\frac{13}{128}\right)\cos(4x)+C.}$$

♦

Example 2. Compute $\displaystyle\int e^{2x}\sin(3x)\,dx$.

Solution. We take $f(x)=\sin(3x)$ and $g(x)=e^{2x}$ and construct the table of derivatives and antiderivatives:

$$
\begin{tikzcd}
u & dv \\
\sin(3x) \arrow[dr, "+"] & e^{2x} \\
3\cos(3x) \arrow[dr, "-"] & \frac{1}{2}e^{2x} \\
-9\sin(3x) \arrow[r, dashed, "+"'] & \frac{1}{4}e^{2x}
\end{tikzcd}
$$

This tells us that

$$\int e^{2x}\sin(3x)\,dx=\frac{1}{2}e^{2x}\sin(3x)-\frac{3}{4}e^{2x}\cos(3x)-\frac{9}{4}\int e^{2x}\sin(3x)\,dx.$$

In other words

$$\frac{13}{4}\int e^{2x}\sin(3x)\,dx=\frac{1}{2}e^{2x}\sin(3x)-\frac{3}{4}e^{2x}\cos(3x)+C$$

so that

$$\int e^{2x}\sin(3x)\,dx = \boxed{\frac{1}{13}e^{2x}\left(2\sin(3x) - 3\cos(3x)\right) + C.}$$

♦

Example 3. Find the 2-periodic cosine expansion of the function $x^2(1-x)$, $0 < x < 1$.

Solution. The half-range Fourier coefficients are given by

$$a_0 = \frac{2}{2\cdot 1}\int_0^1 x^2(1-x)\,dx = \frac{x^3}{3} - \frac{x^4}{4}\bigg|_0^1 = \frac{1}{12}$$

and for $n \geq 1$

$$a_n = \frac{2}{1}\int_0^1 x^2(1-x)\cos(n\pi x)\,dx.$$

We compute the integral here using the tabular technique, with $f(x) = x^2 - x^3$ and $g(x) = \cos(n\pi x)$:

$$\begin{array}{ccc}
u & & dv \\
\hline
x^2 - x^3 & & \cos(n\pi x) \\
& \overset{+}{\searrow} & \\
2x - 3x^2 & & \frac{1}{n\pi}\sin(n\pi x) \\
& \overset{-}{\searrow} & \\
2 - 6x & & \frac{-1}{n^2\pi^2}\cos(n\pi x) \\
& \overset{+}{\searrow} & \\
-6 & & \frac{-1}{n^3\pi^3}\sin(n\pi x) \\
& \overset{-}{\searrow} & \\
0 & \underset{+}{\dashrightarrow} & \frac{1}{n^4\pi^4}\cos(n\pi x)
\end{array}$$

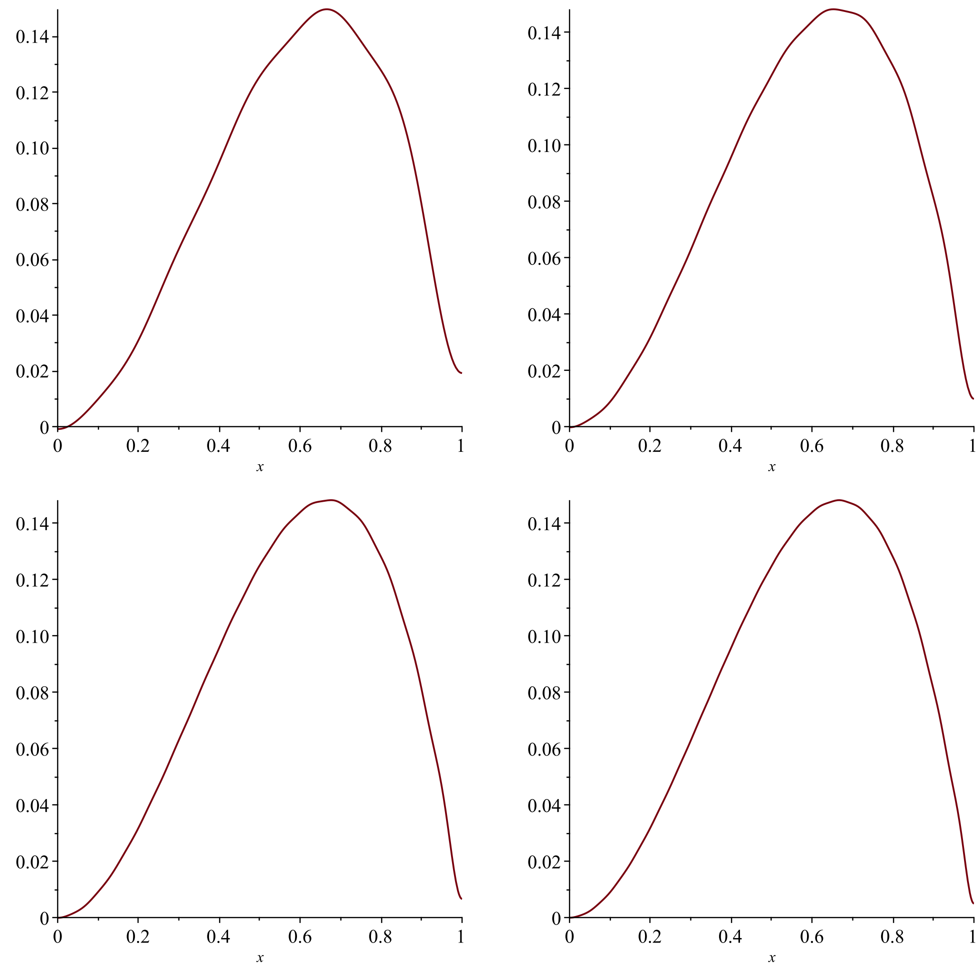
Hence

$$\begin{aligned}
a_n &= 2\left((x^2 - x^3)\frac{\sin(n\pi x)}{n\pi} + (2x - 3x^2)\frac{\cos(n\pi x)}{n^2\pi^2} - (2-6x)\frac{\sin(n\pi x)}{n^3\pi^3} + 6\frac{\cos(n\pi x)}{n^4\pi^4}\right)\Bigg|_0^1 \\
&= 2\left(-\frac{\cos(n\pi)}{n^2\pi^2} + 6\frac{\cos(n\pi)}{n^4\pi^4} - 6\frac{1}{n^4\pi^4}\right) \\
&= 2\frac{(-1)^{n+1}}{n^2\pi^2} + 12\frac{(-1)^n - 1}{n^4\pi^4}.
\end{aligned}$$

Consequently the cosine series is

$$\boxed{\frac{1}{12} + \sum_{n=1}^{\infty}\left(2\frac{(-1)^{n+1}}{n^2\pi^2} + 12\frac{(-1)^n - 1}{n^4\pi^4}\right)\cos(n\pi x).}$$

The plots below show the 10th, 20th, 30th and 40th partial sums of this series on the interval $0 \le x \le 1$, which more closely resemble $f(x) = x^2(1-x)$ as the number of terms increases, as they should.



♦

Example 4. There are numerous situations where repeated integration by parts is called for, but in which the tabular approach must be applied repeatedly. For example, consider the integral

$$\int (\log x)^2 \, dx.$$

If we attempt tabular integration by parts with $f(x) = (\log x)^2$ and $g(x) = 1$ we obtain

$$\begin{array}{cc}
u & dv \\
\hline
(\log x)^2 & 1 \\
 & {\scriptstyle +} \\
\dfrac{2\log x}{x} & \xrightarrow{-} x
\end{array}$$

so that the antiderivative is

$$x(\log x)^2 - 2\int \log x \, dx.$$

There's no point in continuing the table, for if we do so we find that where ever we decide to terminate our columns we will be faced with the antiderivative $\int \log x \, dx$. Evaluating this requires integration by parts. With the choices $f(x) = \log x$ and $g(x) = 1$ we obtain the table

$$\begin{array}{cc} u & dv \\ \hline \log x & 1 \\ & \overset{+}{\searrow} \\ \frac{1}{x} \overset{-}{\dashrightarrow} & x \end{array}$$

so that

$$\int \log x \, dx = x \log x - \int dx = x \log x - x + C.$$

Assembling this with our previous piece we find that

$$\int (\log x)^2 \, dx = \boxed{x(\log x)^2 - 2x \log x + 2x + C.}$$

♦

2 Application: Integrals of the Form $\int P(x)T(\alpha x) \, dx$ where P is a Polynomial and T is Sine or Cosine

Consider an integral of the form $\int P(x)T(\alpha x) \, dx$ where P is a polynomial and T is either sine or cosine. Since $P^{(n)} \equiv 0$ for every sufficiently large n, the corollary to Theorem 1 applies, i.e. the integral term in Theorem 1 reduces to a constant. Moreover, since the derivatives of P are themselves polynomials while the antiderivatives of sine (or cosine) cycle through the functions sine, cosine and their negatives, if we integrate by parts taking $f(x) = P(x)$ and $g(x) = T(\alpha x)$, we find that the antiderivative has the form[4]

$$A(x)\cos(\alpha x) + B(x)\sin(\alpha x) + C \tag{2}$$

where A and B are polynomials. There are two observations that can be made here.

Observation 1: A and B can be computed algebraically, should one wish to avoid integration by parts. First, if we take the derivative of (2) we obtain $(A'(x) + \alpha B(x))\cos(\alpha x) + (-\alpha A(x) + B'(x))\sin(\alpha x)$. Now suppose for the sake of argument that T is cosine.

Then we must have $A' + \alpha B = P$ and $-\alpha A + B' = 0$. Differentiating the first equation, multiplying the second by $-\alpha$ and adding we obtain $A'' + \alpha^2 A = P'$. Assuming α is real, the characteristic equation of the complementary homogeneous ODE $A'' + \alpha^2 A = 0$ has only complex roots. Hence, if we assume A is a polynomial with $\deg A = \deg P'$, we are

[4]This is not the only way to prove this.

guaranteed to find a unique solution to $A'' + \alpha^2 A = P'$ by the method of undetermined coefficients. Once A is in hand, B can be found from the relationship $B = \alpha^{-1}(P - A')$.

It's worth noting, however, that integration by parts is probably far more efficient than the procedure we've just described.

Observation 2: Once we've computed $\int P(x)\sin(\alpha x)\,dx$ either through integration by parts or by using the procedure of the preceding observation, the antiderivative $\int P(x)\cos(\alpha x)\,dx$ can be obtained from the antiderivative $\int P(x)\sin(\alpha x)\,dx$ by simply differentiating every appearance of sine and cosine (formally), i.e. by replacing sine with cosine and cosine with negative sine.

To see this, suppose that

$$\int P(x)\sin(\alpha x)\,dx = A(x)\cos(\alpha x) + B(x)\sin(\alpha x) + C. \tag{3}$$

Perform the formal differentiation of the trigonometric functions to obtain

$$-A(x)\sin(\alpha x) + B(x)\cos(\alpha x) + C. \tag{4}$$

Now derive:

$$(-A'(x) - \alpha B(x))\sin(\alpha x) + (-\alpha A(x) + B'(x))\cos(\alpha x).$$

This doesn't tell us much until we derive equation (3), too:

$$P(x)\sin(\alpha x) = (-\alpha A(x) + B'(x))\sin(\alpha x) + (A'(x) + \alpha B(x))\cos(\alpha x).$$

Comparison of both sides shows that $P(x) = -\alpha A(x) + B'(x)$ and $A'(x) + \alpha B(x) = 0$. Hence the derivative of (4) is exactly $P(x)\cos(\alpha x)$, as claimed. We summarize these findings with the following diagram.

$$\boxed{\begin{tikzcd}
\displaystyle\int P(x)\sin(\alpha x)\,dx \arrow[d, "d"'] & = & A(x)\cos(\alpha x) \arrow[d, "d"'] & + & B(x)\sin(\alpha x) \arrow[d, "d"'] \\
\displaystyle\int P(x)\cos(\alpha x)\,dx & = & -A(x)\sin(\alpha x) & + & B(x)\cos(\alpha x)
\end{tikzcd}}$$