Number Theory | Assignment 12.2
Fall 2020 | Due November 30

Exercise 1. Given a quaternion $z = a + bi + cj + dk \in \mathbb{H}$ with $a, b, c, d \in \mathbb{R}$, recall that $z^* = a - bi - cj - dk$. For $z, w \in \mathbb{H}$ we define their *dot product* and *cross product* to be

$$z \cdot w = \frac{wz^* + zw^*}{2} \quad \text{and} \quad z \times w = \frac{wz^* - zw^*}{2},$$

respectively.

a. Prove that $z \cdot z = a^2 + b^2 + c^2 + d^2$.

b. Prove that if $z, w \in \mathbb{H}$ are purely imaginary (i.e. have no real coordinate), then $z \cdot w$ and $z \times w$ agree with the usual dot and cross products of vectors in $\mathbb{R}^3$.

Exercise 2. Show that if $z \in \mathbb{H}$, then $z^*z = zz^* = N(z)$. Conclude that if $z \neq 0$, then $z^{-1} = z^*/N(z)$.

Exercise 3. Show that every nonzero $z \in \mathbb{H}$ has the *polar form* $z = \rho(\cos\theta + \sin\theta\mathbf{u})$, where $\theta \in \mathbb{R}$, $\rho = \sqrt{N(z)}$, $\mathbf{u}$ is purely imaginary, and $N(\mathbf{u}) = 1$.

Exercise 4. Textbook exercise 13.3.7.

Exercise 5. Textbook exercise 13.3.13.