Base b Expansions of Rational Numbers

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Number Theory

Introduction

As a prelude to a discussion of continued fractions, we consider base b expansions of real numbers.

We will start with the likely familiar fact that every real number possesses a base b expansion

We will then characterize the expansions of rational numbers as those that are eventually periodic, providing an explicit number-theoretic interpretation of the minimal period.

Base b Expansions of Real Numbers

Although likely a "familiar" fact, our first goal is to prove the following fact.

Theorem 1

Let $b \geq 2$ be an integer. For each $x \in [0, 1)$ there exists a sequence $\{d_i\}_{i \geq 1}$ in $\{0, 1, 2, \ldots, b-1\}$ so that

$$x = \sum_{i=1}^{\infty} \frac{d_i}{b^i}.$$

Proof. Let $x \in [0, 1)$, and for each $k \in \mathbb{N}_0$ let $n_k = [b^k x]$, so that

$$n_k \leq b^k x < n_k + 1.$$

If we subtract the $k+1$st inequality from b times the kth we obtain

$$bn_k - (n_{k+1} + 1) < 0 < bn_k + b - n_{k+1} \quad \Rightarrow \quad -1 < n_{k+1} - bn_k < b.$$

Thus $d_{k+1} = n_{k+1} - bn_k \in \{0, 1, 2, \ldots, b-1\}$.

We claim that $\dfrac{n_k}{b^k} = \displaystyle\sum_{i=1}^{k} \frac{d_i}{b^i}$. We proceed by induction on k.

We have $d_1 = n_1 - bn_0 = n_1$, so that $\dfrac{n_1}{b} = \dfrac{d_1}{b}$, which proves the $k = 1$ case.

Now suppose the result is true for some $k \geq 1$. Then

$$\begin{aligned}\frac{n_{k+1}}{b^{k+1}} &= \frac{d_{k+1}}{b^{k+1}} + \frac{bn_k}{b^{k+1}} = \frac{d_{k+1}}{b^{k+1}} + \frac{n_k}{b^k}\\ &= \frac{d_{k+1}}{b^{k+1}} + \sum_{i=1}^{k} \frac{d_i}{b^i} = \sum_{i=1}^{k+1} \frac{d_i}{b^i},\end{aligned}$$

by the inductive hypothesis. This establishes the result for $k+1$, and completes the proof of the claim.

The series $\sum_{i=1}^{\infty} \frac{d_i}{b^i}$ converges by the comparison test, and we've just shown that its kth partial sum is n_k/b^k, which satisfies

$$0 \leq x - \frac{n_k}{b^k} < \frac{1}{b^k}.$$

The squeeze theorem then implies that

$$x = \lim_{k \to \infty} \frac{n_k}{b^k} = \sum_{i=1}^{\infty} \frac{d_i}{b^i}.$$

□

Remark. Note that our proof gives the formula $d_k = [b^k x] - b[b^{k-1} x]$ for computing the "digits" of the base b expansion of $x \in [0, 1)$.

Now let $x \in \mathbb{R}_0^+$ be arbitrary. We can write $x = [x] + (x - [x])$ with $x - [x] \in [0, 1)$ and $[x] \in \mathbb{N}_0$.

Since every member of $\mathbb{N}_0$ has a base b expansion (in nonnegative powers of b), Theorem 2 implies that

$$x = \sum_{i=0}^{n} d_i' b^i + \sum_{i=1}^{\infty} \frac{d_i}{b^i} = [d_n' d_{n-1}' \cdots d_0' . d_1 d_2 d_3 \cdots]_b$$

for some $n \in \mathbb{N}_0$ and $d_i, d_i' \in \{0, 1, 2 \ldots, b - 1\}$.

Remark. The base b expansion of $x < 0$ is obtained by negating the base b expansion of $-x > 0$.

Eventually Periodic Base b Expansions

We say that a base b expansion

$$[d'_n d'_{n-1} \cdots d'_0 . d_1 d_2 d_3 \cdots]_b$$

is *eventually periodic* if there is an $\ell \in \mathbb{N}$ so that $d_{i+\ell} = d_i$ for all sufficiently large i.
This means that we have

$$[d'_n d'_{n-1} \cdots d'_0 . d_1 d_2 d_3 \cdots]_b = [d'_n d'_{n-1} \cdots d'_0 . a_1 a_2 \cdots a_k \overline{c_1 c_2 \cdots c_\ell}]_b,$$

the bar indicating that the string of "digits" $c_1, c_2, \ldots, c_\ell$ is repeated indefinitely.
We say that this expansion is *purely periodic* if $k = 0$, i.e.

$$[d'_n d'_{n-1} \cdots d'_0 . d_1 d_2 d_3 \cdots]_b = [d'_n d'_{n-1} \cdots d'_0 . \overline{c_1 c_2 \cdots c_\ell}]_b.$$

Our first goal is to classify the nonnegative real numbers with purely periodic base b expansions.

If $x \in \mathbb{R}_0^+$ has a purely periodic expansion, then

$$\begin{aligned} x = [d_n' d_{n-1}' \cdots d_0' . \overline{c_1 c_2 \cdots c_\ell}]_b &= n + \sum_{j=0}^{\infty} \sum_{i=1}^{\ell} \frac{c_i}{b^{j\ell+i}} \\ &= n + \sum_{i=1}^{\ell} \sum_{j=0}^{\infty} \frac{c_i}{b^{j\ell+i}} = n + \sum_{i=1}^{\ell} \frac{c_i}{b^i} \sum_{j=0}^{\infty} \frac{1}{b^{j\ell}} \\ &= n + \frac{1}{1 - b^{-\ell}} \sum_{i=1}^{\ell} \frac{c_i}{b^i} = n + \frac{b^\ell}{b^\ell - 1} \sum_{i=1}^{\ell} \frac{c_i}{b^i} \in \mathbb{Q}. \end{aligned}$$

Write $x = r/s$ with $r, s \in \mathbb{N}$ and $(r, s) = 1$. Multiplying the previous equality by $(b^\ell - 1)s$ we find that

$$(b^\ell - 1)r = (b^\ell - 1)sn + b^\ell s \sum_{i=1}^{\ell} \frac{c_i}{b^i} = s\left((b^\ell - 1)n + \sum_{i=1}^{\ell} c_i b^{\ell - i}\right).$$

The quantity on the RHS in parentheses is an integer, so we conclude that $s|(b^\ell - 1)r$.

Since $(r, s) = 1$, Euclid's lemma tells us that $s|b^\ell - 1$ or $b^\ell \equiv 1 \pmod{s}$.

This implies that $(b, s) = 1$ (why?) and that the (multiplicative) order $|b|$ of b modulo s divides ℓ.

To summarize:

Lemma 1

If $(r, s) = 1$ and the base b expansion of r/s is purely periodic with period ℓ, then $(b, s) = 1$ and $m|\ell$, where m is the multiplicative order of b modulo s.

We will now show that if $(r, s) = (b, s) = 1$, then r/s has a purely periodic base b expansion with period equal to the multiplicative order of b modulo s.

Together with Lemma 1 proves that the *minimal* period of the base b expansion of r/s is *precisely* the multiplicative order of b modulo s.

So suppose $(r, s) = (b, s) = 1$ and let m be the multiplicative order of b modulo s.

Write $b^m - 1 = ns$ and $nr = q(b^m - 1) + t$ with $0 \leq t < b^m - 1$.

Then

$$\begin{aligned}
\frac{r}{s} = \frac{nr}{ns} = \frac{q(b^m - 1) + t}{b^m - 1} &= q + \frac{t}{b^m - 1} \\
&= q + \frac{t}{b^m} \frac{1}{1 - b^{-m}} = q + \frac{1}{b^m} \left(\sum_{i=0}^{m-1} d_i b^i \right) \left(\sum_{j=0}^{\infty} \frac{1}{b^{jm}} \right) \\
&= q + \left(\sum_{i=1}^{m} \frac{d_{m-i}}{b^i} \right) \left(\sum_{j=0}^{\infty} \frac{1}{b^{jm}} \right) = q + \sum_{j=0}^{\infty} \sum_{i=1}^{m} \frac{d_{m-i}}{b^{jm+i}} \\
&= \left[q.\overline{d_{m-1} d_{m-2} \cdots d_0} \right]_b .
\end{aligned}$$

Characterization of Purely Periodic Expansions

Thus:

Theorem 2

Suppose $(r, s) = 1$. The base b expansion of r/s is purely periodic with minimal period m if and only if $(b, s) = 1$ and m is the multiplicative order of b modulo s.

Example 1

Verify Theorem 2 for the fraction $2/13$ in base 10.

Solution. Long division yields

$$\frac{2}{13} = 0.\overline{153846},$$

which is purely periodic with minimal period equal to 6.

We have $\varphi(13) = 12 = 2^2 \cdot 3$, so the only possible orders of 10 modulo 13 are 2, 3, 4, 6, 12.

We find that

$$10^2 \equiv 9 \pmod{13}, 10^3 \equiv -1 \pmod{13},$$
$$10^4 \equiv 3 \pmod{13}, 10^6 \equiv 1 \pmod{13},$$

showing that the order of 10 modulo 13 is indeed 6. □

Example 2

Verify Theorem 2 for the purely periodic expansion $3.\overline{11112}$.

Solution. Let $x = 0.\overline{11112}$. Then $10^5 x = 11112 + x$.

Thus

$$x = \frac{11112}{10^5 - 1} = \frac{11112}{99999} = \frac{3704}{33333}.$$

Therefore

$$3.\overline{11112} = 3 + x = 3 + \frac{3704}{33333} = \frac{103703}{33333},$$

and its easy to see that 10 has order 5 modulo 33333. □

We can now easily deal with eventually periodic base b expansions.

p-adic Valuations

Given a prime p and $n \in \mathbb{N}$, let $\nu_p(n) \in \mathbb{N}_0$ denote the exact power of p dividing n.

So, for example, since $12 = 2^2 \cdot 3$,

$$\nu_2(12) = 2, \;\; \nu_3(12) = 1, \;\; \nu_5(12) = 0, \;\; \nu_7(12) = 0, \ldots$$

That is, $n = p^{\nu_p(n)} m$ with $(m, p) = 1$, and

$$n = \prod_p p^{\nu_p(n)},$$

the product extending over all primes p (the product is actually finite since $\nu_p(n) \neq 0$ only for those primes p occurring in the canonical form of n).

Given a base b and $s \in \mathbb{N}$, write

$$s = \underbrace{\prod_{p \nmid b} p^{\nu_p(s)}}_{s'} \underbrace{\prod_{p \mid b} p^{\nu_p(s)}}_{s''},$$

so that $(b, s') = 1$.

Choose n so that $\nu_p(b^n) = n\nu_p(b) \geq \nu_p(s)$ for all $p|b$.

Then $s'' = \prod_{p|b} p^{\nu_p(s)}$ divides b^n: $b^n = s''r'$.

Thus

$$\frac{b^n r}{s} = \frac{s'' r' r}{s' s''} = \frac{r' r}{s'} = \left[a_k a_{k-1} \cdots a_0.\overline{d_1 d_2 \cdots d_m}\right]_b,$$

with minimal period m equal to the multiplicative order of b modulo s', by Theorem 2.

Since multiplication by b^{-n} shifts the "decimal" point to the left by n "digits", we find that r/s has an eventually periodic base b expansion.

Conversely, if r/s (with $(r,s)=1$) has an eventually periodic base b expansion of minimal period m, then for an appropriate $n \in \mathbb{N}$, $b^n r/s$ is purely periodic.

Let $d=(b^n,s)$ and write $b^n = dr'$ and $s = ds'$ with $(r',s')=1$.

Then $b^n r/s = rr'/s'$, with $(rr',s')=1$. Theorem 2 then implies that $(b,s')=1$ and that m is the multiplicative order of b modulo s'.

Eventually Periodic Base b Expansions

Finally, since $s' = s/(b^n, s)$ is relatively prime to b, one can show that

$$s' = \prod_{p \nmid b} p^{\nu_p(s)},$$

as above (HW). This proves our characterization of eventually periodic base b expansions.

Theorem 3

Suppose that $(r, s) = 1$. Then r/s has an eventually periodic base b expansion of minimal period m if and only if m is the multiplicative order of b modulo

$$s' = \prod_{p \nmid b} p^{\nu_p(s)}.$$

Examples

Remark. Note that $(s, b) = 1$ if and only if $s' = s$, so Theorem 3 includes Theorem 2 as a corollary.

Example 3

Verify Theorem 3 for the fraction 791/1850 in base 10.

Solution. Long division yields

$$\frac{791}{1850} = 0.42\overline{756},$$

which is eventually periodic with minimal period 3.

We have

$$s = \underbrace{2 \cdot 5^2}_{s''} \cdot \underbrace{37}_{s'},$$

and

$$10^2 \equiv 26 \pmod{37}, \;\; 10^3 \equiv 1 \pmod{37},$$

so that 10 has order 3 modulo 37, in accordance with Theorem 3. □

Example 4

Verify Theorem 3 for the base 6 expansion $[12.134\overline{45}]_6$.

Solution. Let $x = [.\overline{45}]_6$. Then $6^2 x = 4 \cdot 6 + 5 + x = 29 + x$.

Thus

$$x = \frac{29}{6^2 - 1} = \frac{29}{35}.$$

Hence

$$[12.134\overline{45}]_6 = 6^1 + 2 \cdot 6^0 + \frac{1}{6} + \frac{3}{6^2} + \frac{4}{6^3} + \frac{x}{6^3} = \frac{62539}{7560}.$$

Since $7560 = 6^3 \cdot 35$, we have $s' = 35$ and clearly 6 has order 2 modulo 35. □