Finite Continued Fractions

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Number Theory

Introduction

For us, the theory of continued fractions begins with a reinterpretation of the quotients occurring in the EA.

This will lead to the finite continued fraction representations of rational numbers.

We will eventually show that general (infinite) continued fractions can also be used to represent arbitrary real numbers, and contain useful arithmetic information.

The EA Revisited

Let $x = r/s \in \mathbb{Q}$ with $r, s \in \mathbb{Z}$ and $(r, s) = 1$. The EA yields the sequence of divisions

$$\begin{aligned} r = q_1 s + r_1 &\Rightarrow \frac{r}{s} = q_1 + \frac{r_1}{s}, \\ s = q_2 r_1 + r_2 &\Rightarrow \frac{s}{r_1} = q_2 + \frac{r_2}{r_1}, \\ r_1 = q_3 r_2 + r_3 &\Rightarrow \frac{r_1}{r_2} = q_3 + \frac{r_3}{r_2}, \\ &\vdots \\ r_{n-1} = q_{n+1} r_n &\Rightarrow \frac{r_{n-1}}{r_n} = q_{n+1}, \end{aligned}$$

in which the remainders satisfy

$$|s| > r_1 > r_2 > r_3 > \cdots > r_n = (r, s) = 1.$$

Repeated back substitution yields

$$
\begin{aligned}
\frac{r}{s} &= q_1 + \frac{r_1}{s} \\
&= q_1 + \cfrac{1}{q_2 + \frac{r_2}{r_1}} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \frac{r_3}{r_2}}} \\
&\vdots \\
&= q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{q_4 + \cfrac{1}{\ddots \over q_n + \cfrac{1}{q_{n+1}}}}}}
\end{aligned}
$$

Finite Continued Fractions

An expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \, \cfrac{}{a_{n-1} + \cfrac{1}{a_n}}}}}}$$

is called a *finite (simple) continued fraction*, and we denote it by

$$[a_0; a_1, a_2, \ldots, a_n].$$

We will assume that $a_0 \in \mathbb{R}$ and $a_i \in \mathbb{R}^+$ for $i \geq 1$ (usually in $\mathbb{Z}$).

Rationals as Continued Fractions

Our work above proves:

Theorem 1

Let $r/s \in \mathbb{Q}$ with $(r,s) = 1$. If $q_1, q_2, \ldots, q_{n+1}$ are the quotients appearing the the EA applied to (r,s), then

$$\frac{r}{s} = [q_1; q_2, q_3, \ldots, q_{n+1}].$$

In particular, every rational number can be represented as a finite simple continued fraction.

Example 1

Express $\frac{39}{14}$ as a continued fraction.

Solution. We implement the EA and keep track of the quotients:

$$\begin{aligned} 39 &= \boxed{2} \cdot 14 + 11, \\ 14 &= \boxed{1} \cdot 11 + 3, \\ 11 &= \boxed{3} \cdot 3 + 2, \\ 3 &= \boxed{1} \cdot 2 + 1, \\ 2 &= \boxed{2} \cdot 1. \end{aligned}$$

Therefore

$$\boxed{\frac{39}{14} = [2; 1, 3, 1, 2].}$$

$\square$

Convergents

Definition

Given a finite continued fraction $[a_0; a_1, a_2, \ldots, a_n]$, for $0 \leq k \leq n$ its *kth convergent* is

$$C_k = [a_0; a_1, \ldots, a_k].$$

Example 2

Compute the convergents of $39/14$.

Solution. We have

$$\frac{39}{14} = [2; 1, 3, 1, 2].$$

Therefore:

$$C_0 = [2] = 2,$$

$$
\begin{aligned}
C_1 &= [2;1] = 2 + \frac{1}{2} = 3, \\
C_2 &= [2;1,3] = 2 + \cfrac{1}{1 + \cfrac{1}{3}} = \frac{11}{4}, \\
C_3 &= [2;1,3,1] = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1}}} = \frac{14}{5}, \\
C_4 &= [2;1,3,1,2] = \frac{39}{14}.
\end{aligned}
$$

$\square$

Remark. The convergents of a continued fraction can be unwieldy to deal with directly. Fortunately they can be compute recursively.

Computing Convergents

Given real numbers $a_0, a_1, \ldots, a_n$ with $a_i > 0$ for $i \geq 1$, define two sequences $\{p_k\}_{k=0}^n$ and $\{q_k\}_{k=0}^n$ by

$$\begin{aligned} p_0 &= a_0, & q_0 &= 1, \\ p_1 &= a_1 a_0 + 1, & q_1 &= a_1, \\ p_k &= a_k p_{k-1} + p_{k-2}, & q_k &= a_k q_{k-1} + q_{k-2}, \end{aligned}$$

for $k \geq 2$. We have:

Theorem 2

Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_i > 0$ for $i \geq 1$. Define p_k and q_k as above. Then the convergents of $[a_0; a_1, \ldots, a_n]$ are given by

$$C_k = \frac{p_k}{q_k},$$

for $k \geq 0$.

Proof

We induct on n. Since

$$\frac{p_0}{q_0} = \frac{a_0}{1} = a_0 = C_0, \quad \frac{p_1}{q_1} = \frac{a_1 a_0 + 1}{a_1} = a_0 + \frac{1}{a_1} = C_1,$$

and

$$\begin{aligned}\frac{p_2}{q_2} = \frac{a_2 p_1 + p_0}{a_2 q_1 + q_0} &= \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1} \\ &= a_0 + \frac{a_2}{a_2 a_1 + 1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = C_2,\end{aligned}$$

the result holds for $n = 0, 1, 2$.

Let $n \geq 2$ and assume the result for all sequences $a_0, a_1, \ldots, a_n$.

For $0 \leq k \leq n-1$, the continued fractions

$$[a_0; a_1, \ldots, a_n + 1/a_{n+1}] = [a_0; a_1, \ldots, a_n, a_{n+1}]$$

have the same kth convergents as $[a_0, a_1, \ldots, a_n]$, and the former also has length n.

By the inductive hypothesis its nth convergent is

$$\begin{aligned}
C_{n+1} = [a_0; a_1, \ldots, a_n + 1/a_{n+1}] &= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) q_{n-1} + q_{n-2}} \\
&= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} = \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} \\
&= \frac{p_{n+1}}{q_{n+1}}.
\end{aligned}$$

This completes the induction. □

Example

Consider the continued fraction $\frac{39}{14} = [2; 1, 3, 1, 2]$.

We have $p_0 = 2$, $p_1 = 1 \cdot 2 + 1 = 3$, $p_2 = 3 \cdot 3 + 2 = 11$,

$$p_3 = 1 \cdot 11 + 3 = 14, \quad p_4 = 2 \cdot 14 + 11 = 39,$$

and $q_0 = 1$, $q_1 = 1$, $q_2 = 3 \cdot 1 + 1 = 4$,

$$q_3 = 1 \cdot 4 + 1 = 5, \quad q_4 = 2 \cdot 5 + 4 = 14.$$

These immediately imply that

$$C_0 = 2, \quad C_1 = 3, \quad C_2 = \frac{11}{4}, \quad C_3 = \frac{14}{5}, \quad C_4 = \frac{39}{14},$$

as above. □

When the a_i are integers, the fractions p_k/q_k are always reduced. This is a consequence of the following more specific result.

Theorem 3

If $C_k = p_k/q_k$ is the kth convergent of the continued fraction $[a_0; a_1, \ldots, a_n]$, then

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$$

for $1 \leq k \leq n$

Proof. We induct on k. When $k = 1$ we have

$$p_1 q_0 - q_1 p_0 = (a_1 a_0 + 1) \cdot 1 - a_1 a_0 = 1 = (-1)^0,$$

which is what we needed to show.

Now assume the result for some $1 \leq k < n$. Then $k+1 \geq 2$ so that

$$
\begin{aligned}
p_{k+1}q_k - q_{k+1}p_k &= (a_{k+1}p_k + p_{k-1})q_k - (a_{k+1}q_k + q_{k-1})p_k \\
&= q_k p_{k-1} - p_k q_{k-1} = -(-1)^{k-1} = (-1)^k,
\end{aligned}
$$

proving the that $k+1$ case holds. This completes the proof. $\square$

Corollary 1

If the a_i are integers, then $(p_k, q_k) = 1$ for all $k \geq 1$.

Proof. By Theorem 3 we have

$$(-1)^{k-1} \in p_k\mathbb{Z} + q_k\mathbb{Z} = (p_k, q_k)\mathbb{Z}.$$

$\square$

Theorem 3 has another interesting consequence.

Dividing the equation $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$ through by $q_{k-1}q_k$ we obtain

$$C_k - C_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k-1}}{q_{k-1}q_k}.$$

Replacing k by $k+1$ and adding the resulting equation to the previous, after a little algebra we find that

$$C_{k+1} - C_{k-1} = \frac{(-1)^{k-1}(q_{k+1} - q_{k-1})}{q_{k-1}q_k q_{k+1}}.$$

Because the sequence $\{q_k\}$ is strictly increasing, the sign of the RHS is $(-1)^{k-1}$, which depends only on the parity of k.

If k is odd, we therefore obtain $C_{k-1} < C_{k+1}$, while if k is even we have $C_{k+1} < C_{k-1}$.

It follows that the subsequences $\{C_{2n+1}\}$ and $\{C_{2n}\}$ are strictly decreasing and increasing, respectively.

Finally, if k is even, ℓ is odd and $\ell > k$ we have

$$C_\ell - C_k = C_\ell - C_{\ell-1} + C_{\ell-1} - C_k > 0,$$

while if $\ell < k$

$$C_\ell - C_k = C_\ell - C_{k+1} + C_{k+1} - C_k > 0.$$

In either case we have $C_\ell > C_k$.

Ordering of Convergents

This proves our last result.

Theorem 4

If C_k is the kth convergent of the continued fraction $[a_0; a_1, a_2, \ldots, a_n]$, then

$$C_0 < C_2 < C_4 < \cdots < C_5 < C_3 < C_1.$$

Example. Recall that the convergents of $39/14 = [2; 1, 3, 1, 2]$ are

$$C_0 = 2, \;\; C_1 = 3, \;\; C_2 = \frac{11}{4} = 2.75, \;\; C_3 = \frac{14}{5} = 2.8, \;\; C_4 = \frac{39}{14} \approx 2.786,$$

which satisfy

$$C_0 < C_2 < C_4 < C_3 < C_1.$$

Remarks

If we set

$$p_{-2} = 0, \ \ p_{-1} = 1, \ \ q_{-2} = 1, \ \ q_{-1} = 0,$$

then the relationships

$$p_k = a_k p_{k-1} + p_{k-2} \quad \text{and} \quad q_k = a_k q_{k-1} + q_{k-2}$$

hold for all $k \geq 0$. Therefore, if we let

$$R_k = \begin{pmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{pmatrix} \ \text{ and } A_k = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix},$$

we find that

$$A_k R_{k-1} = \begin{pmatrix} a_k p_{k-1} + p_{k-2} & a_k q_{k-1} + q_{k-2} \\ p_{k-1} & q_{k-1} \end{pmatrix} = R_k.$$

for $k \geq 0$.

It follows that

$$\begin{aligned}
R_k &= A_k R_{k-1} = A_k A_{k-1} R_{k-2} \\
&\vdots \\
&= A_k A_{k-1} \cdots A_0 R_{-1} \\
&= A_k A_{k-1} \cdots A_0,
\end{aligned}$$

since $R_{-1} = I$ by definition. Taking the determinant we immediately obtain

$$p_k q_{k-1} - q_k p_{k-1} = \det R_k = \prod_{i=0}^{k} \det A_i = (-1)^{k+1},$$

which is the conclusion of Theorem 3.