Infinite Continued Fractions

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Number Theory

Introduction

Today we will develop the theory of infinite continued fractions.

By defining an infinite continued fraction to be the limit of its (finite) convergents, we can appeal to many of the results we have already proven.

We will see that infinite continued fractions represent irrational numbers, and conversely that every irrational can be so represented.

Our final result will show that the convergents of an infinite continued fraction yield its "best" rational approximations.

Infinite Continued Fractions

Given a sequence $\{a_i\}_{i\in\mathbb{N}_0}$ of real numbers with $a_i > 0$ for $i > 0$, we define

$$[a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$
$$= \lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n],$$

provided the limit exists.

As before, we call

$$C_n = [a_0; a_1, a_2, \ldots, a_n]$$

the nth *convergent* of the infinite continued fraction $[a_0; a_1, a_2, \ldots]$.

Our results from last time show that if we define $\{p_n\}$ and $\{q_n\}$ by

$$\begin{aligned} p_0 &= a_0, & q_0 &= 1, \\ p_1 &= a_1 a_0 + 1 & q_1 &= a_1, \\ p_n &= a_n p_{n-1} + p_{n-2}, & q_n &= a_n q_{n-1} + q_{n-2}, \end{aligned}$$

for $n \geq 2$ (note that the q_n are positive and strictly increasing for all n), then $C_n = p_n/q_n$ and

$$C_0 < C_2 < C_4 \cdots < C_5 < C_3 < C_1.$$

It follows that

$$\alpha = \lim_{n\to\infty} C_{2n} \quad \text{and} \quad \alpha' = \lim_{n\to\infty} C_{2n+1}$$

both exist and satisfy

$$C_{2n} \leq \alpha \leq \alpha' \leq C_{2n+1} \quad \text{for all} \quad n \geq 0.$$

Thus

$$|\alpha - \alpha'| \leq C_{2n+1} - C_{2n} = \frac{1}{q_{2n+1}q_{2n}} < \frac{1}{q_{2n}^2}.$$

This gives us a convenient convergence criterion for infinite continued fractions.

Theorem 1

Let $\{a_i\}_{i\in\mathbb{N}_0}$ be a sequence of real numbers with $a_i > 0$ for $i > 0$, and define $\{q_n\}$ as above. If $q_n \to \infty$ as $n \to \infty$, then $[a_0; a_1, a_2, a_3, \ldots]$ converges.

Proof. If $q_n \to \infty$, then $1/q_{2n}^2 \to 0$. From the inequality above we therefore have

$$\lim_{n\to\infty} C_{2n} = \alpha = \alpha' = \lim_{n\to\infty} C_{2n+1}.$$

It follows that

$$[a_0; a_1, a_2, a_3, \ldots] = \lim_{n\to\infty} C_n$$

exists. □

Corollary 1

If $a_i \in \mathbb{N}$ for all $i \geq 1$, then $[a_0; a_1, a_2, a_3, \ldots]$ converges.

Proof. Since $q_0 = 1$, $q_1 = a_1$ and

$$q_n = a_n q_{n-1} + q_{n-2} \quad \text{for} \quad n \geq 2,$$

the q_n form an increasing sequence of natural numbers.

Hence $q_n \to \infty$ as $n \to \infty$ and the conclusion follows from Theorem 1. □

Moral. Every integral infinite continued fraction converges!

Example

Example 1

Compute the value of $\alpha = [1; 1, 1, 1, \ldots]$.

Solution. We have

$$\alpha = \lim_{n\to\infty} [1; \underbrace{1, 1, \ldots, 1}_{n \text{ times}}] = \lim_{n\to\infty} 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots + \frac{1}{1}}}$$

$$= 1 + \frac{1}{\lim_{n\to\infty}[1; \underbrace{1, 1, \ldots, 1}_{n-1 \text{ times}}]} = 1 + \frac{1}{\alpha}.$$

This is equivalent to $\alpha^2 - \alpha - 1 = 0$, which we can solve using the quadratic formula.

Indeed,

$$\alpha = \frac{1 \pm \sqrt{5}}{2}.$$

Because α is the limit of positive numbers (its convergents), it cannot be negative.

Therefore

$$\boxed{\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}} = \frac{1 + \sqrt{5}}{2}},$$

which is the *golden ratio*. $\square$

Note that if $\alpha = [a_0; a_1, a_2, \ldots]$ converges, then because it is the limit of both its even-indexed and odd-indexed convergents, we must have

$$C_0 < C_2 < C_4 < \cdots < \alpha < \cdots C_5 < C_3 < C_1.$$

In particular, α is always between C_n and C_{n+1}. This has an interesting consequence.

Theorem 2

If $\alpha = [a_0; a_1, a_2, a_3, \ldots]$ is an integral infinite continued fraction, then α is irrational.

Proof. For any $n \geq 1$ we have

$$0 < |\alpha - C_n| < |C_{n+1} - C_n| = \frac{1}{q_{n+1}q_n}.$$

Assume α is rational: $\alpha = a/b$ with $a, b \in \mathbb{Z}$.

We then have

$$0 < \left| \frac{a}{b} - \frac{p_n}{q_n} \right| < \frac{1}{q_{n+1} q_n} \;\Rightarrow\; 0 < |aq_n - bp_n| < \frac{b}{q_{n+1}}.$$

We have $aq_n - bp_n \in \mathbb{Z}$ for all n, yet $b/q_{n+1} \to 0$ as $n \to \infty$.

This is impossible. $\square$

We have now seen:

- $x \in \mathbb{R}$ is rational iff x is equal to a finite (integral) continued fraction.
- Every (integral) infinite continued fraction is irrational.

It is therefore natural to ask:

Question. Can every irrational (real) number be represented by an integral continued fraction?

Let $x \in \mathbb{R}$ be irrational. Recursively define two sequences $\{x_n\}$ and $\{a_n\}$ as follows.

Set $x_0 = x$ and $a_0 = \lfloor x \rfloor$. Then, given x_n and a_n, define

$$x_{n+1} = \frac{1}{x_n - a_n}, \quad a_{n+1} = \lfloor x_{n+1} \rfloor.$$

Claim. x_n is irrational for all n, $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$.

Proof. We have assumed that $x_0 = x$ is irrational, and $a_0 = \lfloor x \rfloor \in \mathbb{Z}$ by definition.

We prove the remainder of the claim by induction on n. First, we have

$$0 < x_0 - a_0 < 1,$$

since $a_0 = \lfloor x_0 \rfloor$ and $x_0 = x$ is irrational. It follows that

$$x_1 = \frac{1}{x_0 - a_0} > 1$$

is a well-defined irrational number.

Hence $a_1 = \lfloor x_1 \rfloor \in \mathbb{N}$. This proves the $n = 1$ case.

Now assume that x_n is irrational and $a_n \in \mathbb{N}$, for some $n \geq 1$. Then

$$0 < x_n - a_n < 1$$

since x_n is irrational and $a_n = \lfloor x_n \rfloor$. Thus

$$x_{n+1} = \frac{1}{x_n - a_n} > 1$$

is a well-defined irrational number, and

$$a_{n+1} = \lfloor x_{n+1} \rfloor \in \mathbb{N}.$$

This completes the induction. $\square$

Claim. $x = [a_0; a_1, a_2, \ldots, a_n, x_{n+1}]$.

Proof. Again we induct on $n \geq 1$. When $n = 1$ we have

$$[a_0; x_1] = a_0 + \frac{1}{x_1} = a_0 + (x_0 - a_0) = x_0 = x,$$

establishing the $n = 1$ case.

Now assume the result for some $n \geq 1$. Then

$$\begin{aligned}
[a_0; a_1, \ldots, a_n, a_{n+1}, x_{n+2}] &= \left[a_0; a_1, a_2, \ldots, a_{n+1}, \frac{1}{x_{n+1} - a_{n+1}}\right] \\
&= [a_0; a_1, a_2, \ldots, a_{n+1} + (x_{n+1} - a_{n+1})] \\
&= [a_0; a_1, a_2, \ldots, a_n, x_{n+1}] = x,
\end{aligned}$$

and we are finished. $\square$

Now the nth convergent of

$$x = [a_0; a_1, a_2, \ldots, a_n, x_{n+1}] = C'_{n+1}$$

is

$$[a_0; a_1, a_2, \ldots, a_n] = C_n.$$

Thus

$$\begin{aligned} |C_n - x| = |C'_n - C'_{n+1}| &= \frac{1}{(x_{n+1}q_n + q_{n-1})q_n} \\ &< \frac{1}{(a_{n+1}q_n + q_{n-1})q_n} = \frac{1}{q_{n+1}q_n} < \frac{1}{q_n^2}, \end{aligned}$$

since $a_{n+1} < x_{n+1}$.

We have therefore proven:

Theorem 3

Let $x \in \mathbb{R}$ be irrational and define $\{a_n\}$ as above. Then the convergents $C_n = p_n/q_n$ of $[a_0; a_1, a_2, \ldots]$ satisfy

$$|x - C_n| < \frac{1}{q_{n+1}q_n} < \frac{1}{q_n^2}.$$

Corollary 2

Let $x \in \mathbb{R}$ be irrational and define $\{a_n\}$ as above. Then

$$x = [a_0; a_1, a_2, \ldots].$$

Proof. This follows at once since $a_n \in \mathbb{N}$ implies $q_n \to \infty$. $\square$

Remarks

- This result shows that every irrational number is equal to an infinite (integral) continued fraction.
- It is not hard to show that such an expression is unique.
- We can therefore refer to "the" continued fraction expansion of an irrational number.
- Finite continued fraction expansions are only "almost" unique (in a way we won't quantify here).

Examples

Example 2

Compute the continued fraction expansion of $\sqrt{6}$.

Solution. We have

$$x_0 = \sqrt{6}, \quad a_0 = \lfloor \sqrt{6} \rfloor = 2,$$

$$x_1 = \frac{1}{\sqrt{6}-2} = \frac{2+\sqrt{6}}{2} = 1 + \frac{\sqrt{6}}{2}, \quad a_1 = \lfloor x_1 \rfloor = 2,$$

$$x_2 = \frac{1}{1+\frac{\sqrt{6}}{2}-2} = 2+\sqrt{6}, \quad a_2 = \lfloor x_2 \rfloor = 4,$$

$$x_3 = \frac{1}{2+\sqrt{6}-4} = x_1.$$

Because $x_3 = x_1$, the pattern above will continue indefinitely. That is,

$$\sqrt{6} = [2; \overline{2, 4}].$$

Example 3

Find the continued fraction expansion of e.

Solution. With the aid of a computer we find that

$$\begin{aligned}
x_0 &= e, \quad a_0 = \lfloor e \rfloor = 2,\\
x_1 &= \frac{1}{e-2} \approx 1.3922, \quad a_1 = 1,\\
x_2 &= \frac{1}{x_1 - a_1} \approx 2.5496, \quad a_2 = 2,\\
x_3 &= \frac{1}{x_2 - a_2} \approx 1.8193, \quad a_3 = 1,\\
x_4 &= \frac{1}{x_3 - a_3} \approx 1.2204, \quad a_4 = 1,\\
x_5 &= \frac{1}{x_4 - a_4} \approx 4.5355, \quad a_5 = 4,\\
x_6 &= \frac{1}{x_5 - a_5} \approx 1.8671, \quad a_6 = 1.
\end{aligned}$$

One can prove that this pattern persists:

$$\boxed{e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots, 1, 2n, 1, \ldots]}.$$

$\square$

Example 4

Find the first few terms in the continued fraction expansion of π.

Solution. With the aid of a computer we find that

$$\begin{aligned}
x_0 &= \pi, \quad a_0 = \lfloor \pi \rfloor = 3, \\
x_1 &= \frac{1}{e-2} \approx 7.0625, \quad a_1 = 7, \\
x_2 &= \frac{1}{x_1 - a_1} \approx 15.9965, \quad a_2 = 15, \\
x_3 &= \frac{1}{x_2 - a_2} \approx 1.0034, \quad a_3 = 1,
\end{aligned}$$

$$x_4 = \frac{1}{x_3 - a_3} \approx 292.6345, \quad a_4 = 292,$$
$$x_5 = \frac{1}{x_4 - a_4} \approx 1.5758, \quad a_5 = 1.$$

Thus

$$\pi = [3; 7, 15, 1, 292, 1, \ldots],$$

with no (known) pattern. □

Notice that

$$[3; 7] = 3 + \frac{1}{7} = \frac{22}{7},$$

a very well-known approximation to π. This is no coincidence.

One can prove that the convergents of the continued fraction expansion of an irrational number provide the "best" rational approximations, in the following sense.

Theorem 4

Let $x \in \mathbb{R}$ be irrational and let $C_n = p_n/q_n$ be the nth convergent of its continued fraction expansion. If $a, b \in \mathbb{Z}$ and $1 \leq b \leq q_n$, then

$$|x - C_n| \leq \left| x - \frac{a}{b} \right|.$$

Moral. Among all rational numbers with denominator no larger than q_n, C_n is the closest to x.

So, in order to get the "next best" rational approximation to π we need only compute

$$[3;7,15] = 3 + \cfrac{1}{7 + \cfrac{1}{15}} = 3 + \frac{15}{106} = \frac{333}{106} = 3.14150943\ldots$$

There is no better approximation with a denominator ≤ 106.

Have a Great Winter Break!