



PUTNAM SEMINAR  
FALL 2022

QUIZ 11  
DUE NOVEMBER 14

**Problem 1.** Let  $L_1$  and  $L_2$  be distinct lines in the plane. Prove that  $L_1$  and  $L_2$  intersect if and only if, for every real number  $\lambda \neq 0$  and every point  $P$  not on  $L_1$  or  $L_2$ , there exist points  $A_1$  on  $L_1$  and  $A_2$  on  $L_2$  such that  $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$ .

**Problem 2.** A  $2 \times 3$  rectangle has vertices as  $(0, 0)$ ,  $(2, 0)$ ,  $(0, 3)$ , and  $(2, 3)$ . It rotates  $90^\circ$  clockwise about the point  $(2, 0)$ . It then rotates  $90^\circ$  clockwise about the point  $(5, 0)$ , then  $90^\circ$  clockwise about the point  $(7, 0)$ , and finally,  $90^\circ$  clockwise about the point  $(10, 0)$ . (The side originally on the  $x$ -axis is now back on the  $x$ -axis.) Find the area of the region above the  $x$ -axis and below the curve traced out by the point whose initial position is  $(1, 1)$ .

**Problem 3.** Given a point  $(a, b)$  with  $0 < b < a$ , determine the minimum perimeter of a triangle with one vertex at  $(a, b)$ , one on the  $x$ -axis, and one on the line  $y = x$ . You may assume that a triangle of minimum perimeter exists.

**Problem 4.** Let  $P$  be a given (non-degenerate) polyhedron. Prove that there is a constant  $c(P) > 0$  with the following property: If a collection of  $n$  balls whose volumes sum to  $V$  contains the entire surface of  $P$ , then  $n > c(P)/V^2$ .