



PUTNAM SEMINAR
FALL 2022

QUIZ 4
DUE SEPTEMBER 26

Problem 1. Let x_0, x_1, x_2, \dots be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \dots$$

converges and find its sum.

Problem 2. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx}f_{n+1}(x) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

Problem 3. Let $T_0 = 2, T_1 = 3, T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

Problem 4. Let k be a fixed positive integer. The n -th derivative of $\frac{1}{x^k - 1}$ has the form

$$\frac{P_n(x)}{(x^k - 1)^{n+1}}$$
 where $P_n(x)$ is a polynomial. Find $P_n(1)$.