



PUTNAM SEMINAR  
FALL 2022

QUIZ 5  
DUE OCTOBER 3

**Problem 1.** Suppose  $f$  and  $g$  are non-constant, differentiable, real-valued functions defined on  $(-\infty, \infty)$ . Furthermore, suppose that for each pair of real numbers  $x$  and  $y$ ,

$$\begin{aligned}f(x+y) &= f(x)f(y) - g(x)g(y), \\g(x+y) &= f(x)g(y) + g(x)f(y).\end{aligned}$$

If  $f'(0) = 0$ , prove that  $(f(x))^2 + (g(x))^2 = 1$  for all  $x$ .

**Problem 2.** Solve the equations

$$\frac{dy}{dx} = z(y+z)^n, \quad \frac{dz}{dx} = y(y+z)^n,$$

given the initial conditions  $y = 1$  and  $z = 0$  when  $x = 0$ .

**Problem 3.** Suppose that the function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  has continuous partial derivatives and satisfies the equation

$$h(x, y) = a \frac{\partial h}{\partial x}(x, y) + b \frac{\partial h}{\partial y}(x, y)$$

for some constants  $a, b$ . Prove that if there is a constant  $M$  such that  $|h(x, y)| \leq M$  for all  $(x, y) \in \mathbb{R}^2$ , then  $h$  is identically zero.

**Problem 4.** Let  $f$  be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where  $g(x) \geq 0$  for all real  $x$ . Prove that  $|f(x)|$  is bounded.