Putnam Seminar | Quiz 7
Fall 2022 | Due October 17

Problem 1. Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$, where x, y, z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy,$$

and list all such triples T.

Problem 2. Curves A, B, C and D are defined in the plane as follows:

$$A = \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \quad B = \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\},$$
$$C = \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \quad D = \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}.$$

Prove that $A \cap B = C \cap D$.

Problem 3. Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

Problem 4. Let F be a field in which $1 + 1 \neq 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with x and y in F is given by $(x,y) = (1,0)$ and

$$(x,y) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1}\right)$$

where r runs through the elements of F such that $r^2 \neq -1$.