Number Theory — Fall 2023

Assignment 2.3 — Due September 6

Exercise 1. Let $\{f_n\}$ denote the Fibonacci sequence, as defined in Exercise 2.2.2. Let $\beta < \alpha$ denote the roots of the polynomial $x^2 - x - 1$. Use strong induction to prove that

$$f_n = \frac{1}{\sqrt{5}} \left(\alpha^n - \beta^n\right)$$

for all $n \geq 0$. [*Suggestion.* Except in the base cases, *do not* use the expressions for α and β that come from the quadratic formula. Instead, notice that since $\alpha^2 - \alpha - 1 = 0$, we have $\alpha^2 = \alpha + 1$, and likewise for β.]

Exercise 2. If $a, b, n \in \mathbb{N}$, use the strong form of Bézout's lemma to prove that $(a, b) = 1$ if and only if $(a^n, b^n) = 1$. [*Suggestion.* For the forward implication, write $ra + sb = 1$ and expand $(ra + sb)^{2n}$ using the Binomial Theorem. The converse is trivial.]

Exercise 3. Textbook exercise 2.3.20, parts (a)-(e).

Exercise 4. Textbook exercise 2.4.4.