Consider the second order linear ODE

$$ay'' + by' + cy = 0, \tag{1}$$

in which a, b, c are real constants and $a \neq 0$. In class we found the general solution to (1) in a somewhat *ad hoc* manner, by essentially guessing that it might have solutions of the form $y = e^{rx}$, and then appealing to the Principal of Superposition (which we didn't prove) to glue these together into the general solution. Some students find this argument somewhat unsatisfactory, however, since it isn't entirely deductive. It is the goal of this note, therefore, to provide an alternate deductive approach to solving (1).

Dividing both sides of (1) by a and renaming the coefficients yields an equation of the form

$$y'' + by' + cy = 0. \tag{2}$$

The key observation now is to notice that if we "split" b by writing $b = r + s$ for some real numbers r and s, then we can split the middle term by' of (2) to get

$$y'' + by' + cy = y'' + (r+s)y' + cy = y'' + ry' + sy' + cy = (y' + ry)' + (sy' + cy) = 0.$$

The second term in parentheses $(sy' + cy)$ *resembles* the first $(y' + ry)$ multiplied by s. We try to force these terms to be *identical* by setting

$$s(y' + ry) = sy' + cy \iff sry = cy \iff c = sr.$$

That is, if $b = r+s$ and $c = rs$, then $u = y' + ry$ solves the *first order* linear ODE $u' + su = 0$.

Using the integrating factor $I = e^{sx}$, one readily finds that $u = Ce^{-sx}$, where C is an arbitrary constant. Substituting this in $u = y' + ry$ we obtain another *first order* linear ODE in y: $y' + ry = Ce^{-sx}$. Multiplication by the integrating factor $I = e^{rx}$ converts this into

$$(e^{rx}y)' = Ce^{(r-s)x}. \tag{3}$$

If $r \neq s$, integration and subsequent renaming of the constant C yields the general solution

$$y = c_1 e^{-sx} + c_2 e^{-rx}, \tag{4}$$

where c_2 is another arbitrary constant. When $r = s$ equation (3) becomes

$$(e^{rx}y)' = C,$$

and we instead find that

$$y = (c_1 x + c_2)e^{-rx}, \tag{5}$$

where we have renamed C as c_1. The only question that remains is: what are the values of r and s? That is, how can we solve the system $b = r + s$ and $c = rs$ for r and s?

The easiest way to determine r and s is to notice that if $b = r + s$ and $c = rs$, then

$$(X + r)(X + s) = X^2 + (r + s)X + rs = X^2 + bX + c.$$

In other words, r and s are the *negatives* of the roots r_1 and r_2 of the characteristic equation

$$X^2 + bX + c = 0. \tag{6}$$

of (2). Assuming the roots are both real, this means $r = -r_1$, $s = -r_2$, and from (4) and (5) we obtain

$$y = \begin{cases} c_1 e^{r_1 x} + c_2 e^{r_2 x} & \text{when } r_1 \neq r_2, \\ (c_1 x + c_2) e^{r_1 x} & \text{when } r_1 = r_2, \end{cases}$$

in agreement with solutions provided by the Principle of Superposition.

If the roots of (6) happen to be nonreal, then the quadratic formula shows that they must have the form $\alpha \pm i\beta$, where α and β are real and $\beta \neq 0$. Temporarily allowing the solutions of our ODEs to be complex valued, the solution $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ given above is still valid, and appealing to Euler's formula ($e^{i\theta} = \cos\theta + i \sin\theta$) we have

$$\begin{aligned} y &= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) \\ &= e^{\alpha x}\left((c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x\right). \end{aligned}$$

The constants $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$ are now complex, and appear to be linked to one another. But it is easy to see that $c_1 = (C_1 - iC_2)/2$ and $c_2 = (C_1 + iC_2)/2$, so that the values of c_1 and c_2 are determined by C_1 and C_2, and vice versa. So we are free to write the solution in the form

$$y = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right),$$

where C_1 and C_2 are arbitrary (independent) constants which we may choose to be real.

A somewhat more contrived, yet still deductive, argument can be use to treat the case of nonreal complex roots of the characteristic equation. We start by setting $u = ye^{bx/2}$ so that $y = ue^{-bx/2}$. Then the product rule gives

$$\begin{aligned} y' &= \left(u' - \frac{b}{2}u\right) e^{-bx/2}, \\ y'' &= \left(u'' - bu' + \frac{b^2}{4}\right) e^{-bx/2}, \end{aligned}$$

so that

$$y'' + by' + cy = \left(\left(u'' - bu' + \frac{b^2}{4}u\right) + b\left(u' - \frac{b}{2}\right) + cu\right) e^{-bx/2} = \left(u'' - \frac{1}{4}(b^2 - 4c)u\right) e^{-bx/2}.$$

Since the exponential function never vanishes, this shows that the original ODE (1) in y is equivalent to

$$u'' - \frac{1}{4}(b^2 - 4c)u = 0.$$

According to the quadratic formula, the roots of the characteristic polynomial will be complex and nonreal precisely when the discriminant $b^2 - 4ac$ is negative. So we rewrite the ODE above as

$$u'' + \frac{1}{4}(4c - b^2)u = 0 \iff u'' + \delta^2 u = 0,$$

in which the constant $\delta^2 := \frac{1}{4}(4c - b^2)$ is now *positive*. Anticipating the final form of the solution we make another change of variable and set $w = u \sec \delta x$. Then $u = w \cos \delta x$ and the product rule eventually gives us

$$u'' = w'' \cos \delta x - 2\delta w' \sin \delta x - \delta^2 w \cos \delta x,$$

so that

$$u'' + \delta^2 u = w'' \cos \delta x - 2\delta w' \sin \delta x - \delta^2 w \cos \delta x + \delta^2 w \cos \delta x = w'' \cos \delta x - 2\delta w' \sin \delta x.$$

So the ODE $u'' + \delta^2 u = 0$ is equivalent to $w'' \cos \delta x - 2\delta w' \sin \delta x = 0$, which is separable in w'. Indeed, reorganizing we have

$$\frac{w''}{w'} = 2\delta \tan \delta x,$$

and integration gives $\log w' = -2 \log \cos \delta x + C$. Thus $w' = C \sec^2 \delta x$. One more integration tells us that $w = C \tan \delta x + D$ and back substitution gives the penultimate result

$$u = w \cos \delta x = C \sin \delta x + D \cos \delta x.$$

Finally we find that

$$y = e^{-bx/2} u = e^{-bx/2} \left(C \sin \delta x + D \cos \delta x \right).$$

This agrees with the earlier expression for y, since according to the quadratic formula

$$\alpha \pm i\beta = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{-b}{2} \pm i \frac{\sqrt{4c - b^2}}{2} = \frac{-b}{2} \pm i\delta,$$

which shows that $\alpha = -b/2$ and $\beta = \delta$.