Motivating Euler's Formula

R. C. Daileda

Introduction

Euler's formula asserts that for all real θ one has

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1}$$

It provides the rule for evaluating the exponential function at imaginary arguments. But why should we accept this rule? Where does it come from? It is the goal of this note to provide three separate motivations for (1), using only tools from Calculus II. We will treat (1) as the *definition* of $e^{i\theta}$ and will proceed formally (in all cases but the third) in an attempt to justify it.

1 First Order ODEs

Let $E(\theta) = \cos\theta + i\sin\theta$. Notice that $E'(\theta) = -\sin\theta + i\cos\theta = iE(\theta)$. In other words, E solves the first order differential equation

$$\frac{dE}{d\theta} = iE,$$

with initial condition $E(0) = 1$. This initial value problem should be familiar: it is the exponential growth equation with parameter i. In Calculus II one uses separation of variables to show that if k is *real*, then

$$\frac{dy}{dx} = ky \;\;\Rightarrow\;\; y = y_0 e^{kx}.$$

Proceeding by analogy and taking $k = i$, this would mean that $E(\theta) = E_0 e^{i\theta}$. Since $E_0 = E(0) = 1$, we obtain $E(\theta) = e^{i\theta}$, which is (1).

2 Second Order ODEs

The set $\{\cos\theta, \sin\theta\}$ forms a basis for the solution space of the second order homogeneous linear differential equation

$$\frac{d^2y}{d\theta^2} + y = 0, \tag{2}$$

as is easily checked. However, the characteristic equation of (2) is $r^2 + 1 = 0$, with roots $\pm i$. This suggests that the functions e^{itheta} and $e^{-i\theta}$ should also be solutions. In particular, we should be able to express $e^{i\theta}$ as a (complex) linear combination of $\sin\theta$ and $\cos\theta$. So set

$$e^{i\theta} = A\cos\theta + B\sin\theta. \tag{3}$$

Since $e^0 = 1$, we immediately find that $A = 1$. Differentiating (3) and again setting $\theta = 0$ yields $B = 1$, and hence (1) once again.

3 Power Series

In Calculus II one computes the Taylor expansions of the standard elementary functions and finds that for all real x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{4}$$

Likewise (again for all real x),

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{and} \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \tag{5}$$

The power series on the right hand side of (4) converges for all *complex* values of x as well. So if we *define* e^x through equation (4), we immediately find that

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{i^n \theta^n}{n!} \\
&= \sum_{n \text{ even}} \frac{i^n \theta^n}{n!} + \sum_{n \text{ odd}} \frac{i^n \theta^n}{n!} \\
&= \sum_{m=0}^{\infty} \frac{i^{2m} \theta^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i^{2m+1} \theta^{2m+1}}{(2m+1)!} \\
&= \sum_{m=0}^{\infty} \frac{(i^2)^m \theta^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{(i^2)^m i \theta^{2m+1}}{(2m+1)!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m+1}}{(2m+1)!} \\
&= \cos\theta + i \sin\theta,
\end{aligned}$$

by (5). Again we have arrived at (1).

Exercise 1. Where, specifically, does each argument use the fact that $i^2 = -1$?