Complex Variables
Fall 2024

Assignment 13.1
Due December 9

Exercise 1. Prove that the Laurent expansion of an analytic function $f(z)$ with an isolated singularity at z_0 is unique. That is, show that if

$$f(z) = \sum_{n=0}^{\infty} B_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{B_{-n}}{(z-z_0)^n}$$

for $z \in D^*(z_0; r)$, then

$$B_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw$$

for all $n \in \mathbb{Z}$, where C is any positively oriented circle centered at z_0 of radius $R < r$. [*Suggestion.* Substitute the given Laurent expansion into the integral formulas for the "usual" Laurent coefficients A_n, then interchange the order of integration and summation.]

Exercise 2. Let $A \subset \mathbb{C}$ be a domain and let $f : A \to \mathbb{C}$ be analytic. Suppose there is a $z_0 \in A$ so that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}_0$. Show that $f(z) = 0$ for all $z \in A$. [*Suggestion.* Use the "local" version of this result proven in class to show that the set $Z = \{z \in A \,|\, f^{(n)}(z) = 0 \text{ for all } n \in \mathbb{N}_0\}$ is both open and closed, then appeal to the connectivity of A.]

Exercise 3. Let $f(z)$ be analytic on $D^*(z_0; r)$. Show that f has a removable singularity at z_0 if and only if

$$\lim_{z \to z_0} (z - z_0) f(z) = 0.$$

[*Suggestion.* For one implication, use the integral formula for the coefficients in a Laurent expansion, and show that the given condition implies $A_{-n} = 0$ for all $n \in \mathbb{N}$.]

Exercise 4. Use partial fractions to find the Laurent expansion of

$$f(z) = \frac{2z^2 + 13z + 12}{(z+2)^2(z-1)}$$

about the point $z_0 = -2$. What is the largest punctured disk $D(z_0; r)$ on which the Laurent expansion is valid (i.e. equals $f(z)$)?