Exercise 1. Let $X = \mathbb{C} \cup \{\infty\}$. For $\epsilon > 0$, define $D(\infty; \epsilon) = \{z \in \mathbb{C} \mid |z| > \frac{1}{\epsilon}\} \cup \{\infty\}$. We say $U \subseteq X$ is *open* if for every $z \in U$ there exists $\epsilon > 0$ so that $D(z; \epsilon) \subseteq U$.

a. Show that X and $\varnothing$ are both open.

b. If U_α $(\alpha \in I)$ are open subsets of X, show that

$$\bigcup_{\alpha \in I} U_\alpha$$

is also open. Here I is an arbitrary indexing set.

c. If $U_1, U_2, \ldots, U_n$ are open subsets of X, show that

$$\bigcap_{j=1}^{n} U_j$$

is also open.

Parts **a-c** show that the collection of open subsets of X is a *topology* on X. It should be clear that the topology $\mathbb{C}$ inherits as a subspace of X is the usual one.

Exercise 2. Let $f(z)$ be defined on a neighborhood of ∞. Show that $f(1/w)$ is defined on a deleted neighborhood of $w = 0$, and that

$$\lim_{z \to \infty} f(z) = \lim_{w \to 0} f(1/w).$$

Exercise 3. Let $\mathbb{C}[X]$ denote the ring of all polynomials in X with complex coefficients, and for $P(X) \in \mathbb{C}[X]$ let $\deg P$ denote the degree of $P(X)$.

a. Let $P(X) \in \mathbb{C}[X]$ and let $\widetilde{P}(X) = X^{\deg P} P(1/X)$. Prove that if $P(X)$ is nonzero, then $\widetilde{P}(X) \in \mathbb{C}[X]$ and $\widetilde{P}(0)$ is the leading coefficient of $P(X)$.

b. Let $P(X), Q(X) \in \mathbb{C}[X]$ and define $\widetilde{P}(X)$ and $\widetilde{Q}(X)$ as above. Show that

$$\frac{P(1/X)}{Q(1/X)} = X^{\deg Q - \deg P} \frac{\widetilde{P}(X)}{\widetilde{Q}(X)}.$$

c. Let $P(X), Q(X) \in \mathbb{C}[X]$ be nonzero. Use part **b.** and the preceding exercise to show that $\lim_{z\to\infty} \frac{P(z)}{Q(z)}$ exists (in $\mathbb{C}$) if and only if $\deg Q \geq \deg P$.