On the Differentiability of Certain Functions Defined by Path Integrals

R. C. Daileda

Let γ be a piecewise smooth path in $\mathbb{C}$, suppose $g(z)$ is continuous on $\operatorname{im}\gamma$, and let $n \in \mathbb{N}$. Define $f : \mathbb{C} \setminus \operatorname{im}\gamma \to \mathbb{C}$ by

$$f(w) = \int_\gamma \frac{g(z)}{(z-w)^n}\, dz.$$

The purpose of this note is to provide a direct proof of the following result.

Theorem 1. *The function f is analytic on $\mathbb{C} \setminus \operatorname{im}\gamma$ and satisfies*

$$f'(w) = \frac{d}{dw}\int_\gamma \frac{g(z)}{(z-w)^n}\, dz = \int_\gamma \frac{\partial}{\partial w}\frac{g(z)}{(z-w)^n}\, dz = n\int_\gamma \frac{g(z)}{(z-w)^{n+1}}\, dz.$$

Proof. Fix $z_0 \in \mathbb{C} \setminus \operatorname{im}\gamma$. For $w \in \mathbb{C} \setminus \operatorname{im}\gamma$ we have

$$\begin{aligned}
\frac{f(w)-f(z_0)}{w-z_0} &= \frac{1}{w-z_0}\int_\gamma g(z)\left(\frac{1}{(z-w)^n} - \frac{1}{(z-z_0)^n}\right) dz \\
&= \frac{1}{w-z_0}\int_\gamma g(z)\left(\frac{(z-z_0)^n-(z-w)^n}{(z-w)^n(z-z_0)^n}\right) dz \\
&= \frac{1}{w-z_0}\int_\gamma g(z)\left(\frac{\big((z-z_0)-(z-w)\big)\sum_{k=0}^{n-1}(z-z_0)^k(z-w)^{n-1-k}}{(z-w)^n(z-z_0)^n}\right) dz \\
&= \int_\gamma g(z)\sum_{k=0}^{n-1}\frac{1}{(z-z_0)^{n-k}(z-w)^{k+1}}\, dz \\
&= \int_\gamma g(z)\sum_{k=1}^{n}\frac{1}{(z-z_0)^{n-k+1}(z-w)^{k}}\, dz
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{f(w)-f(z_0)}{w-z_0} - n\int_\gamma \frac{g(z)}{(z-z_0)^{n+1}}\, dz &= \int_\gamma g(z)\sum_{k=1}^{n}\left(\frac{1}{(z-z_0)^{n-k+1}(z-w)^k} - \frac{1}{(z-z_0)^{n+1}}\right) dz \\
&= \sum_{k=1}^{n}\int_\gamma g(z)\frac{(z-z_0)^k-(z-w)^k}{(z-z_0)^{n+1}(z-w)^k}\, dz \\
&= \sum_{k=1}^{n}\int_\gamma g(z)\frac{\big((z-z_0)-(z-w)\big)\sum_{\ell=0}^{k-1}(z-z_0)^{k-1-\ell}(z-w)^\ell}{(z-z_0)^{n+1}(z-w)^k}\, dz \\
&= (w-z_0)\sum_{k=1}^{n}\sum_{\ell=0}^{k-1}\int_\gamma \frac{g(z)}{(z-z_0)^{n-k+\ell+2}(z-w)^{k-\ell}}\, dz
\end{aligned}$$

Now let $R = \min\{|z - z_0| : z \in \operatorname{im}\gamma\} > 0$. Then $|z - z_0| \geq R$ for all $z \in \operatorname{im}\gamma$. Moreover, if $|w - z_0| < R/2$, then for any $z \in \operatorname{im}\gamma$ we have

$$|z - w| = |z - z_0 + z_0 - w| \geq |z - z_0| - |z_0 - w| \geq R - \frac{R}{2} = \frac{R}{2}.$$

Thus, for $|w - z_0| < R/2$, along $\operatorname{im}\gamma$ we have

$$\left|\frac{1}{(z - z_0)^{n-k+\ell+2}(z - w)^{k-\ell}}\right| \leq \frac{1}{R^{n-k+\ell+2}(R/2)^{k-\ell}} = \frac{2^{k-\ell}}{R^{n+2}}.$$

Because g is continuous and $\operatorname{im}\gamma$ is compact, there is a constant $M > 0$ so that $|g(z)| \leq M$ for $z \in \operatorname{im}\gamma$. Our work above then implies

$$\begin{aligned}
\left|\frac{f(w) - f(z_0)}{w - z_0} - n\int_\gamma \frac{g(z)}{(z - z_0)^{n+1}}\,dz\right| &\leq |w - z_0|\sum_{k=1}^{n}\sum_{\ell=0}^{k-1}\left|\int_\gamma \frac{g(z)}{(z - z_0)^{n-k+\ell+2}(z - w)^{k-\ell}}\,dz\right| \\
&\leq |w - z_0|\sum_{k=1}^{n}\sum_{\ell=0}^{k-1}\frac{2^{k-\ell}ML(\gamma)}{R^{n+2}} \\
&= |w - z_0|\frac{ML(\gamma)}{R^{n+2}}\sum_{k=1}^{n}\sum_{\ell=0}^{k-1}2^{k-\ell} \\
&= |w - z_0|\frac{ML(\gamma)}{R^{n+2}}\left(2^{n+2} - 2n - 4\right).
\end{aligned}$$

This can be made arbitrarily small by taking w sufficiently close to z_0, proving that

$$f'(z_0) = \lim_{w \to z_0}\frac{f(w) - f(z_0)}{w - z_0} = n\int_\gamma \frac{g(z)}{(z - z_0)^{n+1}}\,dz.$$

Since $z_0 \in \mathbb{C} \setminus \operatorname{im}\gamma$ was arbitrary, the statement of the theorem follows. $\square$

Notice that $f'(w)/n$ has the same form as $f(w)$, but with the exponent $n+1$ in place of n. Because $n \in \mathbb{N}$ was arbitrary in Theorem 1, we can apply the theorem in the $n+1$ case to conclude that $f'(w) = n \cdot f'(w)/n$ is analytic on $\mathbb{C} \setminus \operatorname{im}\gamma$, with

$$f''(w) = \frac{d}{dw}\left(n\int_\gamma \frac{g(z)}{(z - w)^{n+1}}\,dz\right) = n\int_\gamma \frac{\partial}{\partial w}\frac{g(z)}{(z - w)^{n+1}}\,dz = n(n+1)\int_\gamma \frac{g(z)}{(z - w)^{n+2}}\,dz.$$

An inductive argument can be used to show that this line of reasoning continues indefinitely.

Corollary 1. *f is infinitely differentiable on $\mathbb{C} \setminus \operatorname{im}\gamma$. For every $k \geq 0$,*

$$f^{(k)}(w) = \frac{(n + k - 1)!}{(n - 1)!}\int_\gamma \frac{g(z)}{(z - w)^{n+k}}\,dz.$$

In particular, differentiation under the integral sign is valid.