On the Vanishing of Complex Polynomial Functions in z and $\overline{z}$

R. C. Daileda

1 Introduction

The goal of this note is to provide two proofs of the following result.

Theorem 1. *A polynomial of the form*

$$P(z) = \sum_{(m,n)\in\mathbb{N}_0^2} a_{mn} z^m \overline{z}^n, \;\; a_{mn} \in \mathbb{C} \quad \textit{almost all zero,}^1$$

is identically zero on $\mathbb{C}^\times$ *if and only if* $a_{mn} = 0$ *for all* $(m,n) \in \mathbb{N}_0^2$.

Remarks.

1. For polynomials in just z, or in any number of independent variables, the analogous result is relatively easy to prove. For instance, suppose

   $$P(z,w) = \sum_{(m,n)\in\mathbb{N}_0^2} a_{mn} z^m w^n, \;\; a_{mn} \in \mathbb{C} \quad \text{almost all zero,}$$

   is equal to 0 for all $(z,w) \in \mathbb{C}^2$. Notice that

   $$P(z,w) = \sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn} z^m w^n = \sum_{n=0}^{N} \underbrace{\left(\sum_{m=0}^{M} a_{mn} z^m\right)}_{q_n(z)} w^n = \sum_{n=0}^{N} q_n(z) w^n.$$

   For any fixed $z_0 \in \mathbb{C}$ we see that

   $$0 = P(z_0, w) = \sum_{n=0}^{N} q_n(z_0) w^n$$

   for all $w \in \mathbb{C}$. But $P(z_0,w)$ is a polynomial in w with complex coefficients $q_n(z_0)$, which has only finitely many zeros in $\mathbb{C}$ (an impossibility), unless it is the zero polynomial, i.e. all of its coefficients vanish: $q_n(z_0) = 0$ for all n. But $z_0 \in \mathbb{C}$ was arbitrary, so this shows again that $q_n(z)$ is a polynomial with infinitely many roots, making it identically zero. That is, $a_{mn} = 0$ for all m and all n.

   Clearly this argument doesn't apply in our situation. We have $w = \overline{z}$, so that we cannot hold z fixed and vary w as we did above.

[1] That is, $a_{mn} \neq 0$ for only finitely many pairs $(m,n) \in \mathbb{N}_0^2$.

2. The space of functions $f : \mathbb{C}^\times \to \mathbb{C}$ is a complex vector space under pointwise operations. In this context, Theorem 1 asserts that the functions $\chi_{mn}(z) = z^m\overline{z}^n$ are $\mathbb{C}$-linearly independent. Notice that for any $(m,n) \in \mathbb{N}_0^2$ and any $z, w \in \mathbb{C}$ we have

$$\chi_{mn}(zw) = (zw)^m(\overline{zw})^n = z^m w^m \overline{z}^n \overline{w}^n = z^m \overline{z}^m w^m \overline{w}^n = \chi_{mn}(z) \cdot \chi_{mn}(w).$$

As we will see, it is this very property that that is central to our first proof of Theorem 1

2 First Proof

Our first proof of Theorem 1 is purely algebraic. We Let G be a (multiplicative) group. A *character* of G is a homomorphism $\chi : G \to \mathbb{C}^\times$. The set $\mathbb{C}^G$ of functions $f : G \to \mathbb{C}$ is a complex vector space under pointwise operations, and clearly every character of G belongs to $\mathbb{C}^G$. Our main result regarding characters is the following, which is simply Theorem 12 of [1] in the case $F = \mathbb{C}$. Our proof is simply a rephrasing of that given by Artin in [1].

Theorem 2. *If $\chi_1, \chi_2, \dots, \chi_n$ are distinct characters of a group G, then they are $\mathbb{C}$-linearly independent (in $\mathbb{C}^G$).*

Proof. Suppose not. Then there exists a linear combination

$$\sum_{j=1}^{n} a_j \chi_j = 0,$$

with $\mathfrak{a}_j \in \mathbb{C}$ not all zero. If $I = \{1 \le j \le n \,|\, a_j \ne 0\}$, then we may omit the terms with indices outside of I, and the above becomes

$$\sum_{j \in I} a_j \chi_j = 0 \quad \text{with} \quad a_j \ne 0 \quad \text{for all} \quad j \in I. \tag{1}$$

So the collection of nonempty subsets I of $\{1, 2, \dots, n\}$ for which a dependence relation of the type (1) holds is nonempty. Choose I so that $m = |I|$ is as small as possible. Note that $|I| \ge 2$, since $a\chi(g) \ne 0$ for all $a \in \mathbb{C}^\times$ and $g \in G$.

For convenience relabel the characters so that $I = \{1, 2, \dots, m\}$ for some $2 \le m \le n$. We then have

$$a_1\chi_1(g) + \sum_{j=2}^{m} a_j \chi_j(g) = 0 \ \text{(all } a_j \ne 0) \tag{2}$$

for all $g \in G$. If $h \in H$ and we multiply through by $\chi_1(h)$ we obtain

$$a_1\chi_1(gh) + \sum_{j=2}^{m} a_j \chi_j(g)\chi_1(h) = 0.$$

And replacing g in (2) with gh yields

$$a_1\chi_1(gh) + \sum_{j=2}^{m} a_j \chi_j(g)\chi_j(h) = 0.$$

Now if we subtract the second of these two equations from the first we find that

$$\sum_{j=2}^{m} a_j(\chi_1(h) - \chi_j(h))\chi_j(g) = 0,$$

for all $g, h \in G$, or

$$\sum_{j=2}^{m} a_j(\chi_1(h) - \chi_j(h))\chi_j = 0, \tag{3}$$

for all $h \in G$. Since $\chi_1 \neq \chi_2$, we may choose $h \in G$ so that $\chi_1(h) - \chi_2(h) \neq 0$. Then the $j = 2$ coefficient in (3) is nonzero. Omitting those terms for which $\chi_1(h) = \chi_j(h)$ we obtain a sum of the form (1) for which $|I| \leq m - 1 < m$, contradicting the minimality of m. This establishes the theorem. □

Question. Where does the proof use the fact that *all* of the χ_j are pairwise distinct? It would appear that we only need to know that just *two* of them are distinct (χ_1 and χ_2). Why isn't this the case?

This theorem has interesting consequences. For instance, if G is finite of order n, then $\dim \mathbb{C}^G = n$, and Theorem 1 implies that G has at most n distinct characters (exactly n if and only if G is abelian, it turns out). And it should be pointed out that the proof we have given is just as valid if $\mathbb{C}$ is replaced by any other field, which yields Theorem 12 of [1]. It is in this more general context that Artin [1] uses the independence of characters while building up the theorems of Galois theory. Our main application will be to the group $G = \mathbb{C}^\times$ and the characters $\chi_{m,n}(z) = z^m\overline{z}^n$, for $m, n \in \mathbb{Z}$.

First Proof of Theorem 1. It suffices to prove that the characters $\chi_{mn}(z) = z^m\overline{z}^n$ are distinct for distinct pairs $(m, n) \in \mathbb{Z}^2$. To see this, suppose $z^m\overline{z}^n = z^k\overline{z}^\ell$ for all $z \in \mathbb{C}$. Then $z^{m-k} = \overline{z}^{\ell-n}$ for all $z \in \mathbb{C}^\times$. In particular, $2^{m-k} = 2^{\ell-n}$, which implies $m-k = \ell-n = N \in \mathbb{Z}$ by the Fundamental Theorem of Arithmetic. We then have $z^N = \overline{z}^N = \overline{z^N}$ for all $z \in \mathbb{C}^\times$. But this means $z^N \in \mathbb{R}$ for all $z \in \mathbb{C}^\times$, which is impossible (take $z = e^{i\pi/2N}$) unless $N = 0$. Therefore $m = k$ and $n = \ell$, as claimed. As observed above, this completes the proof. □

3 Second Proof

We now turn to our second (analytic) proof, which takes advantage of the differential operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If $A \subseteq \mathbb{C}$ is a nonempty open subset of $\mathbb{C}$, one can show that $\partial/\partial x$ and $\partial/\partial y$ are $\mathbb{C}$-linear on $C^1(A)$, the space of functions $f : A \to \mathbb{C}$ for which $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ have continuous first order partial derivatives. Hence $\partial/\partial\overline{z}$ is $\mathbb{C}$-linear on $C^1(A)$, too. It also obeys the Leibniz product rule:

$$\frac{\partial(fg)}{\partial\overline{z}} = f\frac{\partial g}{\partial\overline{z}} + g\frac{\partial f}{\partial\overline{z}}$$

Moreover, if $f \in C^1(A)$, then f is analytic on A if and only if $\partial f/\partial \overline{z} = 0$, by the Cauchy-Riemann equations.

Notice that

$$\frac{\partial}{\partial \overline{z}}(z^m \overline{z}^n) = z^m \frac{\partial}{\partial \overline{z}}(\overline{z}^n) + \overline{z}^n \frac{\partial}{\partial \overline{z}}(z^m) = z^m \frac{\partial}{\partial \overline{z}}(\overline{z}^n),$$

because of the product rule and the fact that z^m is analytic on $\mathbb{C}^\times$. Furthermore

$$\frac{\partial}{\partial x}\overline{z} = \frac{\partial}{\partial x}(x - iy) = 1 - 0i = 1$$

and

$$\frac{\partial}{\partial x}\overline{z}^{n+1} = \frac{\partial}{\partial x}(\overline{z}^n \cdot \overline{z}) = \overline{z}^n \frac{\partial}{\partial x}\overline{z} + \overline{z}\frac{\partial}{\partial x}\overline{z}^n = \overline{z}^n + \overline{z}\frac{\partial}{\partial x}\overline{z}^n$$

can be used to provide a quick inductive proof of the power rule:

$$\frac{\partial}{\partial x}\overline{z}^n = n\overline{z}^{n-1} \quad \text{for} \quad n \in \mathbb{N}.$$

However,

$$\frac{\partial}{\partial y}\overline{z} = 0 - i = -i$$

so that

$$\frac{\partial}{\partial y}\overline{z}^{n+1} = -i\overline{z}^n + \overline{z}\frac{\partial}{\partial y}\overline{z}^n$$

as above, and the power rule instead becomes

$$\frac{\partial}{\partial y}(z^n) = -in\overline{z}^{n-1} \quad \text{for} \quad n \in \mathbb{N}.$$

We finally conclude that

$$\frac{\partial}{\partial \overline{z}}(\overline{z}^n) = \frac{1}{2}\left(n\overline{z}^{n-1} + i(-in\overline{z}^{n-1})\right) = n\overline{z}^{n-1},$$

for $n \in \mathbb{N}$, as expected. This holds trivially for $n = 0$, provided we interpret $0 \cdot \overline{z}^{-1}$ as the zero function.

We now have

$$\frac{\partial}{\partial \overline{z}}(z^m \overline{z}^n) = z^m \frac{\partial}{\partial \overline{z}}(\overline{z}^n) = nz^m \overline{z}^{n-1}.$$

So if $f(z)$ is an analytic function of z we have

$$\frac{\partial}{\partial \overline{z}}(f(z)\overline{z}^n) = f(z)\frac{\partial}{\partial \overline{z}}\overline{z}^n + \overline{z}^n \frac{\partial}{\partial \overline{z}}f(z) = f(z)\frac{\partial}{\partial \overline{z}}\overline{z}^n = nf(z)\overline{z}^{n-1}. \tag{4}$$

Relative to $\partial/\partial\overline{z}$, analytic functions of z act like constant multiples.

Second Proof of Theorem 1. We prove the contrapositive. Suppose $P(z)$ is *not* identically zero. Then at least one $a_{mn} \neq 0$. That is, there exists an n so that $q_n(z)$ is not the zero polynomial. Let N' denote the largest n so that $q_n(z)$ is not identically zero. Then $a_{mn} = 0$ for all m and $n > N$, and

$$P(z) = \sum_{n=0}^{N} q_n(z)\overline{z}^n = \sum_{n=0}^{N'} q_n(z)\overline{z}^n,$$

so that we may assume $N' = N$ and $q_N(z)$ is not identically zero.

The rule (4), applied N times, tells us that

$$\frac{\partial^N}{\partial \overline{z}^N} P(z) = N! q_N(z),$$

since any term of the form $q_n(z)\overline{z}^n$ with $n < N$ will eventually differentiate to 0 under $\partial/\partial\overline{z}$. Since $q_N(z)$ is not identically zero, neither is $N!q_N(z)$. That is,

$$\frac{\partial^N}{\partial \overline{z}^N} P(z)$$

is not identically zero. But this requires the same to be true of $P(z)$, which is what we needed to show.

□

Remarks.

1. The idea behind this proof comes from an analogous proof for polynomials in one variable. If $p(z) = \sum_{n=0}^{N} a_n z^n$ with $a_n \in \mathbb{C}$, then p and all of its derivatives are complex polynomials in z, and are hence entire. So we may differentiate $p(z)$ as often as we like. We find that

$$p^{(m)}(z) = \frac{d^m}{dz^m} p(z) = \sum_{n=0}^{N} a_n \frac{d^m}{dz^m} z^n = \sum_{m=n}^{N} a_n n(n-1)\cdots(n-(m-1)) z^{n-m},$$

since $\frac{d^m}{dz^m}(z^n) = 0$ if $m > n$. There for $p^{(m)}(0) = a_m \cdot m!$ or

$$a_m = \frac{p^{(m)}(0)}{m!},$$

which should look familiar from the theory of Taylor series. If $p(z)$ is identically 0, then so is $p^{(m)}(z)$ for all $m \in \mathbb{N}_0$, and we find that

$$a_m = \frac{0}{m!} = 0 \quad \text{for all} \quad 0 \leq m \leq N,$$

as expected.

Even though we can write the $P(z)$ from Theorem 1 as

$$P(z) = \sum_{n=0}^{N} q_n(z)\overline{z}^n,$$

where each $q_n(z)$ is a complex polynomial in z, and

$$\frac{\partial^m}{\partial \overline{z}^m} P(z) = \sum_{n=m}^{N} n(n-1)\cdots(n-(m-1))\, q_n(z)\overline{z}^{n-1},$$

we cannot extract the coefficient $q_m(z)$ by simply setting $\overline{z} = 0$ as we did above. Because if $\overline{z} = 0$, then $z = 0$, and all we conclude is that

$$\left.\frac{\partial^m}{\partial \overline{z}^m} P(z)\right|_{z=0} = m!\, q_m(0).$$

So if $P(z)$ is identically zero, all this tells us is that $q_n(0) = 0$ for all n, which doesn't give us much information about the coefficients of $q_n(z)$. To get around this problem, we differentiated $P(z)$ N times, leaving *only* $N! q_N(z)$ as the Nth derivative. Therefore, from the vanishing of $P(z)$, we get $N! q_N(z) = 0$ for all $z \in \mathbb{C}$, which makes $q_n(z)$ the zero polynomial. This reduces the $\overline{z}$-"degree" of $P(z)$ by one, as we have seen, and allows us to get our hands on the remaining $q_n(z)$ by repeating this argument (that is, by induction).

2. My original second proof was along the same lines, but used induction on N. I'd like to thank Arseny Mingajev for pointing out the shorter proof given here.

References

[1] Artin, E., *Galois Theory*, Dover Publications (1998).