Putnam Exam Seminar
Fall 2010

Assignment 4
Due October 4

Exercise 1. The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n. [Putnam Exam, 1999, A-6]

Exercise 2. Let $\{x_n\}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n \geq 1$. Prove that there is a real number a so that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$. [Putnam Exam, 1993, A-2]

Exercise 3. Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

[Putnam Exam, 1999, A-3]

Exercise 4. Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

[Putnam Exam, 2004, B-5]