Exercise 1. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh},\ f(0) = 1, \\ g' &= fg^2h + \frac{4}{fh},\ g(0) = 1, \\ h' &= 3fgh^2 + \frac{1}{fg},\ h(0) = 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0. [Putnam Exam, 2009, A-2]

Exercise 2. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes. [Putnam Exam, 1985, B-2]

Exercise 3. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$. [Putnam Exam, 2005, B-3]

Exercise 4. Let f be a real function on the real line with a continuous third derivative. Prove that there exists a point a such that $f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0$. [Putnam Exam, 1998, A-3]

Exercise 5. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}} \left(\frac{p(x)}{x^3 - x} \right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$. [Putnam Exam, 1992, B-4]