Putnam Exam Seminar
Fall 2010

Assignment 9
Due November 8

Exercise 1. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically? [Putnam Exam, 2003, B1]

Exercise 2. Let n be a positive integer and define

$$f(n) = 1! + 2! + 3! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

Exercise 3. Let $P(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_nr, c_nr^2 + c_{n-1}r, c_nr^3 + c_{n-1}r^2 + c_{n-1}r, \ldots, c_nr^n + c_{n-1}r^{n-1} + \cdots c_1r$$

are integers. [Putnam Exam, 2004, B1]

Exercise 4. For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients? [Putnam Exam, 2001, A3]

Exercise 5. Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z)$ have absolute value 1. [Putnam Exam, 2005, A3]