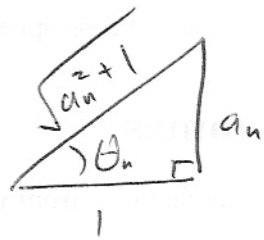
<u>Problem</u>: Define $a_1 = 1$ and $a_{n+1} = \cos(\arctan(a_n))$ for all $n \geq 1$. Find a formula for a_n and determine $\lim_{n\to\infty} a_n$.

<u>Solution</u>: We have $a_n = \sqrt{f_n / f_{n+1}}$ where f_n is the n^{th} Fibonacci number, defined by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. Also

$$\lim_{n\to\infty} a_n = \sqrt{\frac{\sqrt{5}-1}{2}}.$$

We begin by defining $\theta_n = \arctan a_n$. Then, appealing to the right triangle



(That this is valid requires us to prove $0 < \theta_n < \pi/2$. We postpone this for now.)

we find that

$$a_{n+1}^2 = \frac{1}{a_n^2 + 1}$$

for $n \geq 1$. We now prove inductively that $a_n = \sqrt{f_n / f_{n+1}}$. When $n = 1$ we have $a_1 = 1 = \sqrt{1/1} = \sqrt{f_1 / f_2}$. So, now assume $a_n = \sqrt{f_n / f_{n+1}}$ for some $n \geq 1$.

Then

$$a_{n+1}^2 = \frac{1}{a_n^2+1} = \frac{1}{f_n/f_{n+1}+1} = \frac{f_{n+1}}{f_n+f_{n+1}} = \frac{f_{n+1}}{f_{n+2}}$$

so that

$$a_{n+1} = \sqrt{\frac{f_{n+1}}{f_{n+2}}}$$

as desired. By induction, this formula holds for all $n \geq 1$.

Since it is well-known that $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}$ (the golden ratio) we see immediately that

$$\lim_{n\to\infty} a_n = \sqrt{\frac{2}{1+\sqrt{5}}} = \sqrt{\frac{2(1-\sqrt{5})}{1-5}} = \sqrt{\frac{\sqrt{5}-1}{2}}.$$

It remains only to show that $\theta_n = \arctan a_n$ satisfies $0 < \theta_n < \pi/2$. This we also do by induction. When $n=1$, $a_n = 1$ so $\theta_n = \arctan a_n = \pi/4$, which certainly satisfies $0 < \theta_1 < \pi/2$. Now suppose we know $0 < \theta_n < \pi/2$ for some $n \geq 1$. Then $0 < \cos\theta_n < 1$ so that $0 < a_{n+1} < 1$. But then $\theta_{n+1} = \arctan a_{n+1}$ satisfies $0 < \theta_{n+1} < \pi/4 < \pi/2$, since $\arctan 0 = 0$ and arctan is an increasing function on $[0, \pi/2)$. So, by induction, the needed result holds.