Problem: A function f is defined for all positive integers and satisfies

$$f(1) = 2010$$

and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n).$$

Compute $f(2010)$ exactly.

Solution: $f(2010) = \dfrac{2}{2011}$.

First, if $n \geq 2$ we have

$$\begin{aligned} n^2 f(n) &= f(n) + f(n-1) + \cdots + f(2) + f(1) \\ &= f(n) + (n-1)^2 f(n-1) \end{aligned}$$

so that

$$f(n) = \frac{(n-1)^2}{n^2-1} f(n-1) = \frac{n-1}{n+1} f(n-1).$$

We claim now that $f(n) = \dfrac{4020}{(n+1)n}$, which we prove by induction. When $n=1$ we have

$$\frac{4020}{(n+1)n} = \frac{4020}{2} = 2010 = f(1).$$

Now assume $f(n) = \dfrac{4020}{(n+1)n}$ for some $n \geq 1$. Then

$$f(n+1) = \frac{(n+1)-1}{(n+1)+1} f((n+1)-1) = \frac{n}{n+2} f(n)$$

$$= \frac{n}{n+2} \cdot \frac{4020}{(n+1)n}$$

$$= \frac{4020}{(n+1)(n+2)} = \frac{4020}{((n+1)+1)(n+1)}$$

which is exactly what we needed to show. By induction we conclude our equation holds for all $n \geq 1$. Therefore

$$f(2010) = \frac{4020}{(2011)(2010)} = \frac{2}{2011}.$$