Putnam Exam Seminar — Quiz 11

Fall 2010 — December 1

Problem 1. A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c? [Putnam Exam, 2009, B2]

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for all real numbers x, y and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) - g(y)$ for all real numbers x and y. [Putnam Exam, 2008, A1]

Problem 3. Let f be a nonconstant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$. [Putnam Exam, 2007, B1]

Problem 4. Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points A, B and C which are the vertices of an equilateral triangle, and find its area. [Putnam Exam, 2006, B1]

Problem 5. Define a function f on the real numbers by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/q & \text{if } x = p/q \text{ with } p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(p,q) = 1. \end{cases}$$

Determine the set of points on which f is continuous.

Problem 6. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.) [Putnam Exam, 2004, A3]