Putnam Exam Seminar — Quiz 5

Fall 2010 — October 18

Problem 1. Find the unique function $u(t)$ so that

$$u'(t) = u(t) + \int_0^1 u(s)\,ds$$

and $u(0) = 1$. [Putnam Exam, 1958, 3]

Problem 2. If

$$\begin{aligned} u &= 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots\,, \\ v &= x + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots\,, \\ w &= \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots\,, \end{aligned}$$

then prove that

$$u^3 + v^3 + w^3 - 3uvw = 1.$$

[Putnam Exam, 1939, 14]

Problem 3. For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

[Putnam Exam, 1987, A-3]

Problem 4. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$ determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b). [Putnam Exam, 1988, A-2]