Math 1311-1 Fall 2006

Calculus I

Second Midterm Exam - Practice Problems

Problem 1. Maximize the area of a right triangle with fixed hypotenuse length L.

Problem 2. Find the points on the ellipse with equation $x^2 + 4y^2 = 5$ whose tangent lines have x-intercept -5.

Problem 3. Find an equation for the tangent line to the *lemniscate* $2(x^2+y^2)^2 = 25(x^2-y^2)$ at the point $(3, 1)$.

Problem 4. A man standing at the edge of a pier is pulling his boat in from the sea by a rope attached to the bow of the boat. The pier is 2 m higher than the bow of the boat and, as he pulls, the man keeps the rope 1 m above the surface of the pier. If the man is pulling in rope at a rate of 1 m/s, how fast is the boat approaching the pier when it is 8 m from the pier?

Problem 5. Consider a rectangle with fixed height h and width $2h$. An ellipse is circumscribed about the rectangle so that its axes are parallel to the the sides of the rectangle and so that the 4 corners of the rectangle all lie on the ellipse. Find the minimum area (in terms of h) of such an ellipse. You may find it useful to know that the area of the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.

Problem 6.

(a) Use the linear approximation to show that for small x

$$(1 + x)^k \approx 1 + kx.$$

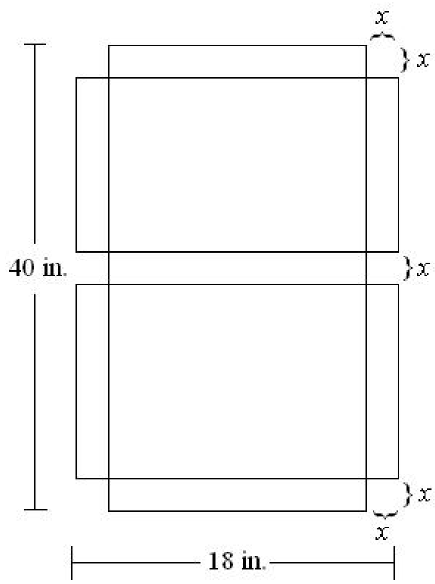
(b) Use part (a) to estimate 1.01^{100}.

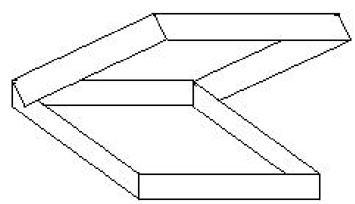
(c) Use part (a) to estimate 0.9^{100}. Is this approximation reasonable?

Problem 7. Find the maximum and minimum values of the function $f(x) = \sin^{2/3} x$ on the interval $[-3\pi/4, 3\pi/4]$.

Problem 8. Show that if $f'(x) \neq 1$ for all real x, then the equation $f(x) = x$ has at most one solution.

Problem 9. A box with an attached lid is to be made from a rectangular piece of cardboard 18 in. wide and 40 in. tall by first cutting out 6 small squares as shown below and then folding up the resulting flaps. What are the dimensions of the box with largest possible volume?





Problem 10. If

$$f(x) = \begin{cases} \sqrt{x} & , \text{ if } x \geq 0 \\ -\sqrt{-x} & , \text{ if } x < 0 \end{cases}$$

then the solution of the equation $f(x) = 0$ is $x = 0$. Explain why Newton's method fails fo find the root no matter which initial approximation $x_0 \neq 0$ is used. Illustrate your explanation with a sketch.

Problem 11. Find the maximum possible area of a right triangle if the lengths of its legs add to 10.

Problem 12. If two resistors with resistances R_1 and R_2 are wired in parallel, the resulting net resistance R is determined by the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Estimate the change in R if R_1 is increased from 2 to 2.1 and R_2 is increased from 4 to 4.2.

Problem 13. What are the dimensions of the right circular cylinder with surface area 2 ft^2 and the largest possible volume?

Problem 14. Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

Problem 15. A particle is traveling along the curve $(x^2 + y^2)^2 = -4x^2y$. At the instant that it passes through the point $(1, -1)$, the x-coordinate of the particle is increasing at 3 units per second. What is the rate of change of the particle's y-coordinate at this instant?

Problem 16. Find the maximum and minimum values of the function $g(x) = (x-1)^2e^{-x}$ on the interval $[2, 4]$.

Problem 17. Use logarithmic differentiation to prove that for any constant c

$$\frac{d}{dx}x^c = cx^{c-1}.$$

Problem 18. Find the derivatives of the following functions.

(a)
$$y = x^{\ln x}$$

(b)
$$y = (\ln x)^x$$

(c)
$$y = 5^{-1/x}$$

(d)
$$y = \sqrt{x}e^{x^2}(x^2+1)^{10}$$

Problem 19. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

Problem 20.

(a) Use the definition of the derivative to prove that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

(b) Apply the natural exponential function to both sides of the above and conclude that

$$\lim_{x\to 0}(1+x)^{1/x} = e.$$

Problem 21. Explain why Newton's method doesn't work for finding the root of the equation $x^3 - 3x + 6 = 0$ if the initial approximation is chosen to be $x_0 = 1$.

Problem 22. Show that for small x we have

$$\sin x \approx x.$$

Problem 23. Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real solution.

Problem 24. Find all points on the curve $x^2y^2 + xy = 2$ where the slope of the tangent line is -1.

Problem 25. Show that $\tan x > x$ for $0 < x < \pi/2$. [*Hint:* Show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$].

Problem 26. Use Newton's method with initial approximation $x_0 = 1$ to find the next approximation x_1 to the root of the equation $x^4 - x - 1 = 0$. Use a sketch including the graph of the function and its tangent line, to explain how x_1 is found.

Problem 27. Find the intervals of increase or decrease for the following functions.

(a)
$$f(x) = x + \cos x, \ -2\pi \le x \le 2\pi$$

(b)
$$g(x) = x\sqrt{x+3}$$

(c)
$$h(x) = (x^2 - 1)^3$$

Problem 28. [**Challenge**] A rectangle is drawn in the first quadrant of the xy-plane so that its base rests on the x-axis and its upper corners both lie on the curve $y = x - x^3$. What is the maximum area of such a rectangle? [*Hint:* Let h denote the height of the rectangle and let r and s denote the two positive solutions to the equation $x - x^3 = h$. Express the area of the rectangle in terms of r.]