Calculus I
Spring 2008

Final assignment

Instructions: The following exercises constitute the essay portion of the final exam for Math 1311. You should write up your solutions neatly and carefully, paying particular attention to explanation of your work in *WORDS* as well as mathematical symbols. Your responses do not need to be typed and there is no minimum length requirement, however handwritten solutions need to be perfectly legible and your solutions must be as complete as possible. You are to complete this assignment *on your own*; the only person from whom you may seek assistance is Professor Daileda; the only references you may consult are the textbook and your lecture notes from the course. Any deviation from these guidelines will constitute a violation of the honor code. If you have any questions, don't hesitate to ask!

Exercise 1. Before Newton's and Leibniz's co-discovery of Calculus (indeed, before they were born!), the seventeenth century mathematician Fermat had developed his own techniques for computing equations of tangent lines to curves. Descartes challenged Fermat to find the tangent to the curve $x^3 + y^3 = 3axy$ (the *Folium of Descartes*) at the point $(3a/2, 3a/2)$, and predicted that Fermat would be unable to do so. Use Calculus to find an equation for this tangent line. Explain how Calculus takes all of the "challenge" out of this problem.

Exercise 2. Snell's law (discovered experimentally in 1621) states that if a light ray passes from a medium in which the speed of light is v_1 into a medium in which the speed of light is v_2, with angle of incidence α and angle of refraction β (see the figure below), then

$$\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}.$$

(i.e. the ratio of the sines of the angles of incidence and refraction is, for fixed media, constant). This can be derived from *Fermat's principle of least time*: a ray of light will take the path that minimizes the total travel time. Begin by writing the time T it takes for the ray to travel from P to Q in terms of the variable x and the constants a, b, v_1, v_2 and s shown in the figure. Then use Calculus to explain why T is minimized precisely when Snell's law holds.

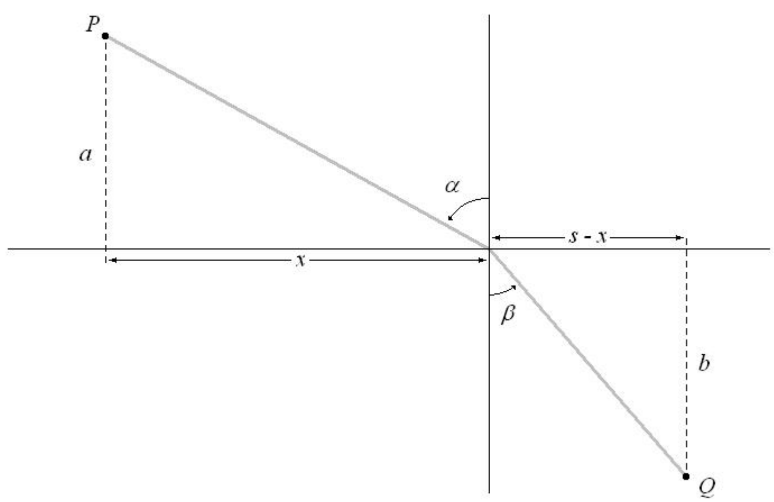

Figure 1: Diagram illustrating the various quantities involved in Snell's law. The horizontal line represents tha boundary between the two media: above this line the speed of light is v_1 and below it is v_2.

Exercise 3. The figure below shows an example of a *parabolic segment*: the bounded region of the plane cut off by a parabola and a line. If we assume the parabola is given by an equation of the form $y = ax^2 + bx + c$, then the point C is the point on the parabola whose x coordinate is halfway between those of A and B. In *Quadrature of the Parabola*, written in the 3rd century B.C., Archimedes showed (without Calculus) that the area of a parabolic segment is always four-thirds the area of the triangle ABC. This fact allows one to compute the area of a parabolic segment from the area of the associated triangle. Explain, in *words and symbols*, how the area of a parabolic segment can, instead, easily be computed using Calculus.

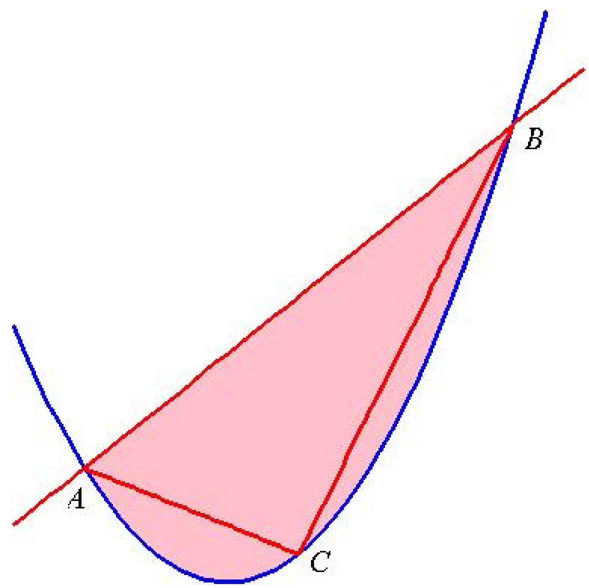

Figure 2: A parabolic segment and its associated triangle.