On the Solutions to Second Order ODEs

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Consider the second order differential equation

$$ay'' + by' + cy = 0 \tag{1}$$

where a, b, c are real constants and $a \neq 0$. By a *solution* to (1) we mean a function $y = f(x)$ defined everywhere on the real line that is twice differentiable and whose derivatives satisfy the relationship specified by (1). In order to find the most general solution (i.e. complete list of solutions) to this differential equation, we introduced the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

and found that by analyzing its solutions we could produce all of the solutions to (1), thereby reducing an apparently complicated calculus problem to a rather simple algebraic computation. A bit more specifically, we used the solutions to (2) to write down two linearly independent functions y_1 and y_2 that we then verified directly were solutions to (1). To produce *every* solution from these two we made an appeal to the following "deep" result.

Theorem 1. *If y_1 and y_2 are linearly independent solutions to* (1) *then every function of the form*

$$y = c_1 y_1 + c_2 y_2,$$

where c_1 and c_2 are any real numbers, is also a solution. Moreover, every *solution of* (1) *has this form.*

We call this result "deep" because its proof is beyond the scope of our class. While the theorem very conveniently provides us with a way of extracting a complete list of solutions from only two of them, it may seem somewhat unsatisfactory to rely on a result whose demonstration is beyond our reach.

The goal of this note is to bridge this gap in our knowledge. We won't prove Theorem 1, but we will prove the next result, which while not as general, still provides us with all of the solutions to (1).

Theorem 2. *If the characteristic equation* (2) *has two distinct real roots r_1 and r_2 then the general solution to* (1) *is given by*

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{3}$$

where c_1 and c_2 are arbitrary constants. If the characteristic equation (2) *has only a single (repeated) real root r then the general solution to* (1) *is given by*

$$y = (c_1 x + c_2) e^{rx} \tag{4}$$

where c_1 and c_2 are arbitrary constants. And if the characteristic equation (2) *has nonreal (complex) roots $\alpha \pm \beta i$ then the general solution to* (1) *is given by*

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \tag{5}$$

where c_1 and c_2 are arbitrary constants.

Proof. It is easy (but perhaps a bit tedious) to check that the functions given in (3) to (5) do, indeed, yield solutions to (1) in each case. The question is: are these the only solutions? We show in every case that they are.

We can deal with the first two cases simultaneously. Suppose that y is a solution to (1) and that r_1 is a real root of (2). Define a new function u by

$$u = ay' + (ar_1 + b)y. \tag{6}$$

Since y and y' are differentiable so is u, and in fact

$$u' = ay'' + (ar_1 + b)y'.$$

Therefore

$$\begin{aligned} u' - r_1 u &= ay'' + (ar_1 + b)y' - r_1 ay' - r_1(ar_1 + b)y \\ &= ay'' + by' - (ar_1^2 + br_1)y. \end{aligned}$$

However, we know that $ar_1^2 + br_1 + c = 0$ so that $ar_1^2 + br_1 = -c$. That is

$$u' - r_1 u = ay'' + by' - (ar_1^2 + br_1)y = ay'' + by' + cy = 0.$$

Therefore, u is a solution to the *first order* linear differential equation $u' - r_1 u = 0$! Solving this using our earlier techniques we find that we must have $u = C_1 e^{r_1 x}$ for some constant C_1. If we put this back into (6) we find that

$$ay' + (ar_1 + b)y = C_1 e^{r_1 x}$$

which is another first order linear equation, this time in y! Dividing by a (which we know to be nonzero) it becomes

$$y' + \left(r_1 + \frac{b}{a}\right) y = C_2 e^{r_1 x} \tag{7}$$

where $C_2 = C_1/a$.

If r_2 is the other root of $ar^2 + br + c = 0$ (which may, in fact, equal r_1), then the polynomial $ax^2 + bx + c$ must factor as $a(x - r_1)(x - r_2)$. If we multiply out the latter polynomial and compare its x coefficient with the first, we find that $b = -a(r_1 + r_2)$, or $-r_2 = r_1 + b/a$.[1] So we can rewrite (7) as

$$y' - r_2 y = C_2 e^{r_1 x}.$$

Using the integrating factor $e^{-r_2 x}$ and solving we find that y must be given by

$$y = C_2 e^{r_2 x} \int e^{(r_1 - r_2)x}\, dx.$$

There are now two cases. If $r_1 \neq r_2$ (i.e. the characteristic equation has two distinct real solutions) then $r_1 - r_2 \neq 0$ and so we have

$$\int e^{(r_1 - r_2)x}\, dx = \frac{e^{(r_1 - r_2)x}}{r_1 - r_2} + C_3 \tag{8}$$

which means that

$$y = C_2 e^{r_2 x} \left(\frac{e^{(r_1 - r_2)x}}{r_1 - r_2} + C_3 \right) = C_4 e^{r_1 x} + C_3 e^{r_2 x},$$

where we have set $C_4 = C_2/(r_1 - r_2)$. Up to the names of the constants (which are arbitrary anyway), this is exactly what we needed! But what if $r_1 = r_2$ so that (8) isn't valid? In this case we have

$$y = C_2 e^{r_2 x} \int e^{(r_1 - r_2)x}\, dx = C_2 e^{r_1 x} \int dx = C_2 e^{r_1 x}(x + C_3) = (C_2 x + C_4)e^{r_1 x},$$

with $C_4 = C_2 C_3$, which gives us what we expected in this case, too.

Now let's move on to the third case, in which the characteristic equation (2) has nonreal roots $\alpha \pm \beta i$. Since we're after *real*-valued solutions in a situation which more naturally calls for complex numbers, things are a bit

[1] This can also be seen directly by appealing to the quadratic formula, which expresses r_1 and r_2 in terms of a, b and c.

more complicated. The first thing we need is a relationship between α, β and the coefficients of (2). As above, knowing the roots of a polynomial allows us to factor it, so that $ax^2+bx+c = a(x-(\alpha+\beta i))(x-(\alpha-\beta i)) = a(x^2-2\alpha x+(\alpha^2+\beta^2))$. Comparing coefficients in these expressions tells us that we must have

$$\begin{aligned} b &= -2a\alpha && (9) \\ c &= a(\alpha^2+\beta^2). && (10) \end{aligned}$$

We'll need these relationships shortly.

Now assume that y is a solution to (1). We're going to perform a series of changes of variables to get (1) into a friendlier form. We first set

$$u = e^{-\alpha x} y. \tag{11}$$

Differentiating twice yields

$$u'' = e^{-\alpha x}\left(y'' - 2\alpha y' + \alpha^2 y\right)$$

so that

$$\begin{aligned} au'' + \beta^2 au &= e^{-\alpha x}\left(ay'' - 2a\alpha y' + a(\alpha^2+\beta^2)y\right) \\ &= e^{-\alpha x}\left(ay'' + by' + cy\right) \\ &= 0 \end{aligned}$$

where we have used (9), (10) and the fact that y solves (1). If we divide both sides of this equation by a, which we know to be nonzero, we obtain the simple linear equation

$$u'' + \beta^2 u = 0. \tag{12}$$

At this point it's worth remembering that our independent variable has been assumed to be x. We now make the substitution $t = \beta x$. According to the chain rule

$$\begin{aligned} u' &= \frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \beta\frac{du}{dt} \\ u'' &= \frac{d^2u}{dx} = \frac{d}{dx}\frac{du}{dx} = \frac{d}{dx}\left(\beta\frac{du}{dt}\right) = \beta\frac{d^2u}{dt^2}\frac{dt}{dx} = \beta^2\frac{d^2u}{dt^2}. \end{aligned}$$

Therefore (12) becomes

$$\beta^2\frac{d^2u}{dt^2} + \beta^2 u = 0.$$

Since we know $\alpha+\beta i$ is definitely not real it must be that $\beta \neq 0$. We can thus divide both sides of the equation above by β^2 which tells us that

$$\frac{d^2u}{dt^2} + u = 0. \tag{13}$$

To solve (13) for u we perform one final substitution, letting

$$w = \frac{du}{dt} + (\tan t)u. \tag{14}$$

Because $\tan t$ is only defined on intervals of the form $I_n = (n\pi - \pi/2, n\pi + \pi/2)$, where n is an integer, w is only defined on these intervals. So, from this point on let's assume that our t domain is a single I_n. We find that

$$\begin{aligned} \frac{dw}{dt} - (\tan t)w &= \frac{d^2u}{dt^2} + (\sec^2 t)u + \tan t\frac{du}{dt} - \tan t\frac{du}{dt} - (\tan^2 t)u \\ &= \frac{d^2u}{dt^2} + (\sec^2 t - \tan^2 t)u \\ &= \frac{d^2u}{dt^2} + u \\ &= 0. \end{aligned}$$

That is, w satisfies the linear equation $dw/dt - (\tan t)w = 0$, which is solved easily using the integrating factor $\cos t$. In fact

$$w = C_1 \sec t \tag{15}$$

for some constant C_1. Referring back to (14), this means that

$$\frac{du}{dt} + (\tan t)u = C_1 \sec t,$$

which is once again linear. The integrating factor $\sec t$ allows us to finally obtain

$$u = C_1 \sin t + C_2 \cos t$$

for some constant C_2.

At this point the back substitutions $t = \beta x$ and $y = e^{\alpha x} u$ tell us immediately that

$$y = e^{\alpha x} \left(C_2 \cos \beta x + C_1 \sin \beta x \right), \tag{16}$$

and we're finished. Well, not quite. There's still one technical point we need to address. When we defined w in the previous paragraph we had to assume that its independent variable t was restricted to the interval I_n. This means that the expression for y in (16) is only valid on these intervals, and that in principle the constants C_1 and C_2 might vary as we vary n. However, u and du/dt are continuous (since they are both differentiable), and we can take limits at the endpoints of each interval to find that the constants match everywhere. For example, if we have

$$\begin{aligned} u &= A_1 \cos t + B_1 \sin t \text{ for } t \in I_0 \\ u &= A_2 \cos t + B_2 \sin t \text{ for } t \in I_1 \end{aligned}$$

then

$$B_1 = \lim_{t \to \pi/2^-} (A_1 \cos t + B_1 \sin t) = u\left(\frac{\pi}{2}\right) = \lim_{t \to \pi/2^+} (A_2 \cos t + B_2 \sin t) = B_2$$

and a similar computation with du/dt gives $A_1 = A_2$. The general case is left to the reader. This completes the proof of the third case. $\square$

The method of proof we've just employed actually still works (in principle) if we modify (1) so that it is no longer homogeneous. That is, if we replace (1) with

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is nonzero, the substitution $u = ay' + (ar_1 + b)y$ of the proof (in the case of real roots) yields the first order differential equation

$$u' - r_1 u = G(x)$$

for u, which is simply the inhomogeneous version of $u' - r_1 u = 0$. This equation is still linear, and if we solve it we can back substitute into $u = ay' + (ar_1 + b)y$ and solve for y. The reader is encouraged to give this a try in the cases where $G(x)$ is a constant or an exponential function.