On the Solutions to Certain Second Order ODEs

R. C. Daileda

Consider the second order differential equation

$$ay'' + by' + cy = 0 \tag{1}$$

where a, b, c are real constants and $a \neq 0$. In our course (and indeed in every course involving equations of this type that I have ever taught) the general solution to (1) is "found" as follows. One first defines the *characteristic equation* of (1) to be

$$ar^2 + br + c = 0 \tag{2}$$

and then easily verifies that if r is a real root of the characteristic equation then $y = e^{rx}$ is a solution to (1) (see the course text or lecture notes). Then, using certain theorems on general second order linear differential equations, one concludes that the general solution to (1) can be constructed from these exponential solutions. While this approach may be perfectly satisfactory to the seasoned mathematician, the invocation of mysterious theorems whose proofs are "too difficult to present here" may leave the average student somewhat frustrated. It is the goal, therefore, of this note to show that it is possible, in an elementary way, to solve (1) directly, i.e. produce the general solution using nothing more than standard results from Calculus. In fact, we will prove the following result.

Theorem 1. *If the characteristic equation (2) has two distinct real roots r and $\widetilde{r}$ then the general solution to (1) is given by*

$$y = c_1 e^{rx} + c_2 e^{\widetilde{r}x} \tag{3}$$

where c_1 and c_2 are arbitrary constants. If the characteristic equation (2) has only a single (repeated) real root r then the general solution to (1) is given by

$$y = c_1 x e^{rx} + c_2 e^{rx} \tag{4}$$

where c_1 and c_2 are arbitrary constants.

Proof. It is easy to check (again, see the course text or lecture notes) that the given functions do, indeed, yield solutions to (1) in each case. The question is: are these the only solutions? We show that they are.

Suppose that $y(x)$ is a function that is twice differentiable at every point of the real line and that satisfies (1), i.e. that $y(x)$ is a solution to (1). Let r be a real root of (2). Assuming that $r \neq 0$ (we will deal with the case $r = 0$ later), we can write (1) in the form

$$\left(ay' - \frac{c}{r}y\right)' + \left(b + \frac{c}{r}\right)y' + cy = 0$$

or, equivalently

$$\frac{d}{dx}\left(ay' - \frac{c}{r}y\right) = -\left(\left(b + \frac{c}{r}\right)y' + cy\right). \tag{5}$$

Notice that since $ar^2 + br + c = 0$ we have

$$\begin{aligned} ar^2 &= -(br + c) \\ ar &= -\frac{br + c}{r} \\ ar &= -\left(b + \frac{c}{r}\right) \end{aligned}$$

so that (5) is the same as

$$\frac{d}{dx}\left(ay' - \frac{c}{r}y\right) = -ary' + cy = r\left(ay' - \frac{c}{r}y\right).$$

This shows that $u = ay' - cy/r$ is a solution to the first order equation

$$\frac{du}{dx} = ru.$$

Since the solutions to this equation are all of the form $u = c_1e^{rx}$, we conclude that

$$ay' - \frac{c}{r}y = c_1e^{rx}$$

for some constant c_1. Since $a \neq 0$ we can divide by it and rewrite this last result as

$$\frac{dy}{dx} - \frac{c}{ar}y = c_1e^{rx}$$

(as usual, since c_1 is an unspecified constant, we replace c_1/a with c_1 again). We now reflect on the constant c/ar appearing here. Using the quadratic formula we have

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

so that

$$\begin{aligned}
\frac{c}{ar} &= \frac{2c}{-b + \sqrt{b^2 - 4ac}} \\
&= \left(\frac{2c}{-b + \sqrt{b^2 - 4ac}}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}\right) \\
&= \frac{2c(-b - \sqrt{b^2 - 4ac})}{b^2 - (b^2 - 4ac)} \\
&= \frac{-b - \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

which is just $\widetilde{r}$, the *other* root of the characteristic equation (2). So now we must solve

$$\frac{dy}{dx} - \widetilde{r}y = c_1e^{rx} \tag{6}$$

which is just a first order linear equation! As usual, we can solve this equation by multiplying by the factor $\rho(x) = e^{-\widetilde{r}x}$ and integrating. This yields

$$e^{-\widetilde{r}x}\frac{dy}{dx} - \widetilde{r}e^{-\widetilde{r}x}y = c_1e^{(r-\widetilde{r})x}$$

or

$$\frac{d}{dx}\left(e^{-\widetilde{r}x}y\right) = c_1e^{(r-\widetilde{r})x} \tag{7}$$

If $r \neq \widetilde{r}$ (i.e. the characteristic equation (2) has two distinct real roots) then integrating yields

$$e^{-\widetilde{r}x}y = c_1e^{(r-\widetilde{r})x} + c_2$$

(here, as above, we have rewritten $c_1/(r - \widetilde{r})$ as c_1 again) or

$$y = c_1e^{rx} + c_2e^{\widetilde{r}x}$$

which is the desired solution (3). If $r = \widetilde{r}$ (i.e. the characteristic equation (2) has a single, repeated, root) then the right hand side of (7) is just c_1 and integrating gives

$$e^{-\widetilde{r}x}y = c_1x + c_2$$

or, since $r = \widetilde{r}$,

$$y = c_1 x e^{rx} + c_2 e^{rx}$$

which is exactly (4).

Finally we address the case $r = 0$. This can only occur if $c = 0$, in which case (1) can be written

$$\frac{d}{dx}\left(ay' + by\right) = 0$$

which immediately gives

$$ay' + by = c_1$$

or

$$\frac{dy}{dx} + \frac{b}{a}y = c_1 \tag{8}$$

(once again absorbing a into the constant c_1). But $c = 0$ implies that the other root of the characteristic equation (2) is $\widetilde{r} = -b/a$ and so (8) can be written

$$\frac{dy}{dx} - \widetilde{r}y = c_1 = c_1 e^{rx}$$

which is just equation (6). We can then complete the proof exactly as above. □

The differential equation (1) is called *homogeneous* because of the zero appearing on the right hand side. It may be worth noting that the method of proof of Theorem 1 can, in certain cases, be used to solve *inhomogeneous* equations. For example, the reader should have no trouble solving inhomogeneous equations of the form

$$ay'' + by' + cy = f(x)$$

where $f(x)$ is a constant or a simple exponential function.