Calculus III
Spring 2009

Exam 3 Practice Problems

Problem 1. Prove the following extension of the Fundamental Theorem of Calculus to two variables: If $\dfrac{\partial^2 F}{\partial x \partial y} = f(x,y)$, and $f(x,y)$ is continuous, then

$$\iint_R f(x,y)\,dA = F(b,d) - F(a,d) + F(a,c) - F(b,c)$$

where $R = [a,b] \times [c,d]$.

Problem 2. Find a function $F(x,y)$ satisfying $\dfrac{\partial^2 F}{\partial x \partial y} = 6x^2 y$ and use the result of the preceding exercise to evaluate $\iint_R 6x^2 y\,dA$ for the rectangle $R = [0,1] \times [0,4]$.

Problem 3. Evaluate $\displaystyle\int_1^3 \int_0^1 y e^{xy}\,dy\,dx$.

Problem 4. Sketch the region R inside the circle $r = 4\cos\theta$ and outside the circle $r = 1$ and find its area. [*Ans:* $7\arccos(1/4) + \sqrt{15}/2$]

Problem 5. Let E be the region in $\mathbb{R}^3$ bounded by $z = 4 - y^2$, $y = 2x$, $z = 0$ and $x - 0$. Express the integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral in all 6 possible orders.

Problem 6. Evaluate $\iiint_E x^2\,dV$ where E is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Problem 7. Find the volume of the region above the plane $z = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

Problem 8. Find a linear transformation $T(u,v)$ that carries the square $S = [0,1] \times [0,1]$ in the uv-plane to the square R in the xy-plane with vertices $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$. Use this transformation to evaluate $\iint_R xy\,dA$.

Problem 9. Let R be the region in the first quadrant enclosed by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 5$, $xy = 1$ and $xy = 3$. Evaluate $\iint_R x^2 + y^2 \, dA$. [*Hint*: See the second extra credit problem.]

Problem 10.[Extra Credit] Find the volume of the region enclosed by the three cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$.

Problem 11.[Extra Credit] Consider the transformation $(x, y) = T(u, v)$ whose inverse is given by $(u, v) = T^{-1}(x, y) = (x^2 - y^2, 2xy)$. Show that $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{4\sqrt{u^2 + v^2}}$.