Recall that in class we defined the linear approximation to a function \( f(x,y) \) of two variables at the point \((a,b)\) to be the function

\[
L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)
\]

and that we defined \( f \) to be differentiable at \((a,b)\) provided

\[
\lim_{(x,y) \to (a,b)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.
\]

The motivation for this definition came from Calculus I, in which case a similar limit holds for differentiable functions of a single variable, and one way to interpret it is that the tangent plane becomes a very good approximation to the graph of \( f \) as \((x,y)\) gets closer and closer to \((a,b)\). In this note we’ll solve two exercises from Stewart’s book which show that two specific functions are differentiable anywhere.

**Exercise 43.** Show that the function \( f(x,y) = x^2 + y^2 \) is differentiable everywhere.

**Solution.** Let \((a,b)\) be an arbitrary point in \(\mathbb{R}^2\). Since \(f(a,b) = a^2 + b^2\), \(f_x(a,b) = 2a\) and \(f_y(a,b) = 2b\) we find that the linear approximation to \( f \) at \((a,b)\) is

\[
L(x,y) = a^2 + b^2 + 2a(x-a) + 2b(x-b) = 2ax + 2by - a^2 - b^2.
\]

Therefore

\[
f(x,y) - L(x,y) = x^2 + y^2 - 2ax - 2by + a^2 + b^2 = (x-a)^2 + (y-b)^2.
\]

It follows that

\[
\lim_{(x,y) \to (a,b)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = \lim_{(x,y) \to (a,b)} \frac{(x-a)^2 + (y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}} = \lim_{(x,y) \to (a,b)} \frac{(x-a)^2 + (y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}} = 0,
\]

since the final function to the right of the limit symbol is continuous at \((a,b)\). Since the necessary limit is zero, \( f \) is differentiable at \((a,b)\).

**Exercise 44.** Show that the function \( f(x,y) = xy - 5y^2 \) is differentiable anywhere.

**Solution.** Let \((a,b)\) be an arbitrary point in \(\mathbb{R}^2\). Since \(f(a,b) = ab - 5b^2\), \(f_x(a,b) = b\) and \(f_y(a,b) = a - 10b\) we find that the linear approximation to \( f \) at \((a,b)\) is

\[
L(x,y) = ab - 5b^2 + b(x-a) + (a-10b)(x-b) = bx + (a-10b)y - ab + 5b^2.
\]
Therefore
\[ f(x, y) - L(x, y) = xy - 5y^2 - bx - ay + 10by + ab + 5b^2 \]
and so we have
\[
\left| \frac{f(x, y) - L(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| = \left| \frac{(x-a)(y-b) - 5(y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}} \right|
\]
\[
\leq \left| \frac{(x-a)(y-b) + 5(y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}} \right|
\]
\[
\leq \left| \frac{(x-a)(y-b)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| + \frac{5(y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}}
\]
\[
= \frac{1}{2} \left( \frac{(x-a)^2 + (y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}} \right) + \frac{5(y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}}
\]
\[
= \frac{1}{2} \sqrt{(x-a)^2 + (y-b)^2} + 5|y-b|.
\]
Since both \( \sqrt{(x-a)^2 + (y-b)^2} \) and \( |y-b| \) tend to 0 as \((x, y) \to (a, b)\), the squeeze law implies that
\[
\lim_{(x,y)\to(a,b)} \frac{f(x, y) - L(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0
\]
which means that \( f \) is differentiable at \((a, b)\).