Recall that in class we defined the linear approximation to a function $f(x,y)$ of two variables at the point (a,b) to be the function

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

and that we defined f to be differentiable at (a,b) provided

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0.$$

The motivation for this definition came from Calculus I, in which case a similar limit holds for differentiable functions of a single variable, and one way to interpret it is that the tangent plane becomes a very good approximation to the graph of f as (x,y) gets closer and closer to (a,b). In this note we'll solve two exercises from Stewart's book which show that two specific functions are differentiable anywhere.

Exercise 43. Show that the function $f(x,y) = x^2+y^2$ is differentiable everywhere.

Solution. Let (a,b) be an arbitrary point in $\mathbb{R}^2$. Since $f(a,b) = a^2+b^2$, $f_x(a,b) = 2a$ and $f_y(a,b) = 2b$ we find that the linear approximation to f at (a,b) is

$$L(x,y) = a^2+b^2+2a(x-a)+2b(x-b) = 2ax+2by-a^2-b^2.$$

Therefore

$$f(x,y)-L(x,y) = x^2+y^2-2ax-2by+a^2+b^2 = (x-a)^2+(y-b)^2.$$

It follows that

$$\begin{aligned} \lim_{(x,y)\to(a,b)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} &= \lim_{(x,y)\to(a,b)} \frac{(x-a)^2+(y-b)^2}{\sqrt{(x-a)^2+(y-b)^2}} \\ &= \lim_{(x,y)\to(a,b)} \sqrt{(x-a)^2+(y-b)^2} \\ &= 0 \end{aligned}$$

since the final function to the right of the limit symbol is continuous at (a,b). Since the necessary limit is zero, f is differentiable at (a,b).

Exercise 44. Show that the function $f(x,y) = xy-5y^2$ is differentiable anywhere.

Solution. Let (a,b) be an arbitrary point in $\mathbb{R}^2$. Since $f(a,b) = ab-5b^2$, $f_x(a,b) = b$ and $f_y(a,b) = a-10b$ we find that the linear approximation to f at (a,b) is

$$L(x,y) = ab-5b^2+b(x-a)+(a-10b)(x-b) = bx+(a-10b)y-ab+5b^2.$$

Therefore

$$\begin{aligned} f(x,y) - L(x,y) &= xy - 5y^2 - bx - ay + 10by + ab + 5b^2 \\ &= (x-a)(y-b) - 5(y-b)^2 \end{aligned}$$

and so we have

$$\begin{aligned} \left| \frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} \right| &= \left| \frac{(x-a)(y-b) - 5(y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}} \right| \\ &= \frac{|(x-a)(y-b) - 5(y-b)^2)|}{\sqrt{(x-a)^2 + (y-b)^2}} \\ &\leq \frac{|(x-a)(y-b) + 5(y-b)^2|}{\sqrt{(x-a)^2 + (y-b)^2}} \\ &\leq \frac{|(x-a)(y-b)| + 5(y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}} \\ &= \frac{|(x-a)(y-b)|}{\sqrt{(x-a)^2 + (y-b)^2}} + \frac{5(y-b)^2}{\sqrt{(x-a)^2 + (y-b)^2}} \\ &\leq \frac{\frac{1}{2}((x-a)^2 + (y-b)^2)}{\sqrt{(x-a)^2 + (y-b)^2}} + \frac{5(y-b)^2}{\sqrt{(y-b)^2}} \\ &= \frac{1}{2}\sqrt{(x-a)^2 + (y-b)^2} + 5|y-b|. \end{aligned}$$

Since both $\sqrt{(x-a)^2 + (y-b)^2}$ and $|y-b|$ tend to 0 as $(x,y) \to (a,b)$, the squeeze law implies that

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

which means that f is differentiable at (a,b).