Introduction to Abstract Mathematics
Spring 2008
Review Problems for Exam 2

Review Problem 1: Let G, H be groups and let $\phi : G \to H$ be a function.

a. If we wanted to show that G is Abelian, what should the first line of the proof be.

b. If we wanted to show that ϕ is one-to-one, what should the first line of the proof be.

c. If we wanted to show that ϕ is onto, what should the first line of the proof be.

d. If we wanted to show that ϕ is operation preserving, what should the first line of the proof be.

Review Problem 2: Define a binary operation on a set G.

Review Problem 3: Define a group.

Review Problem 4: Define what it means for a group to be Abelian.

Review Problem 5: State Cayley's theorem.

Review Problem 6: State the Fundamental Theorem of Cyclic groups.

Review Problem 7: State the Division Algorithm.

Review Problem 8: Given a group G with subgroup H of G, we define the set

$$xHx^{-1} = \{xhx^{-1} \mid h \in H\}.$$

We then define the *normalizer of* H to be

$$N(H) = \{x \in G \mid xHx^{-1} = H\}.$$

Show that $N(H) < G$.

Review Problem 9: Let G be a finite group and suppose $x, y, z \in G$.

a. Show that $\mid xy \mid = \mid yx \mid$.

b. Show that it is not necessarily the case that $\mid xyz \mid = \mid zyx \mid$.

Review Problem 10: Is there an infinite group where each element has finite order? Justify.

Review Problem 11: Is there a finite group with an element of infinite order?

Review Problem 12: Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is a group. Assume that the binary operation is modular addition on each component.

Review Problem 13: Find all subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$.

Review Problem 14: Is $\mathbb{Z}_3 \times \mathbb{Z}_4$ cyclic? How about $\mathbb{Z}_4 \times \mathbb{Z}_6$?

Review Problem 15:

(a) Is $(\mathbb{Z}, \cdot)$ a group?

(b) Find all subsets $X \subset \mathbb{Z}$ so that $(X, \cdot)$ is a group.

(c) If X is as in part (b), is $(X, \cdot) < (\mathbb{Z}, +)$?

Review Problem 16: Show that isomorphism is an equivalence relation on the collection of groups.

Review Problem 17: Let G be an Abelian group and let $a \in G$. Given $x, y \in G$ define

$$x \times y := xya^{-1}.$$

a. Prove that $(G, \times)$ is a group.

b. Prove that $(G, \times)$ is isomorphic to G with its original operation.

Review Problem 18: Let G be an Abelian group with identity e and let $n \in \mathbb{N}$. Let $G[n] = \{x \in G \,|\, x^n = e\}$. Prove that $G[n] < G$.

Review Problem 19: Find all subgroups of $\mathbb{Z}$ that contain both 6 and 15.

Review Problem 20: Let G be a group and suppose that $a_1, a_2, \ldots a_n \in G$. Show that

$$(a_1 a_2 \cdots a_n)^{-1} = a_n^{-1} \cdots a_2^{-1} a_1^{-1}.$$