Math 2326 - Introduction to Abstract Mathematics
Review Problems for Exam 3

Problem 1: State the definition of a countable set.

Problem 2: State the Archimedean Principle.

Problem 3: State the Axiom of Completeness for $\mathbb{R}$.

Problem 4: State the definition of *supremum* and *infimum* of a bounded set $A \subseteq \mathbb{R}$.

Problem 5: State the Nested Interval Property.

Problem 6: State the definition of an ε-neighborhood of $a \in \mathbb{R}$.

Problem 7: State the definition of an *open* set.

Problem 8: State Cantor's Theorem.

Problem 9: Show that $\{x \in \mathbb{R} |\ x \in \mathbb{Q} \text{ or } x^2 \in \mathbb{Q}\}$ is countable.

Problem 10: Show that the set of all quadratic polynomials with rational coefficients is countable.

Problem 11: Let A, B be nonempty subsets of $\mathbb{R}$ such that for all $x \in A$ and $y \in B$, we have that $x \leq y$. Show that $\sup A \leq \inf B$. Furthermore, show that $\sup A = \inf B$ if and only if for all $\varepsilon > 0$, we may find $x \in A$ and $y \in B$ such that $y - x < \varepsilon$.

Problem 12: Given $x \in \mathbb{R}$, let $S_x = \{q \in \mathbb{Q} |\ q \leq x\}$. Show that $\sup S_x = x$.

Problem 13: Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded. Let $A \cdot B = \{x \cdot y |\ x \in A, y \in B\}$. Show that

a) $A \cdot B$ is bounded

b) $\sup(A \cdot B) = (\sup A)(\sup B)$

c) $\inf(A \cdot B) = (\inf A)(\inf B)$

Problem 14: Show that $\lim \frac{1}{\sqrt{n}} = 0$.

Problem 15: Show that $\lim(-1)^n$ does not exist.

Problem 16: Show that the Axiom of Completeness holds for $\mathbb{Z}$.

Problem 17: Show that the Axiom of Completeness does not hold for $\mathbb{Q}$.

Problem 18: Let $\{x_n\}$ be a sequence of real numbers.

a) Show that if $\{x_n\}$ is convergent then it is bounded.

b) State the contrapositive statement of the previous item.

c) Using the previous items, describe the heuristics to prove the following: If $x_n \neq 0$ and $\lim \frac{x_{n+1}}{x_n} = L > 1$, then $\{x_n\}$ diverges.

Problem 19: Let $\{y_n\}$ be a bounded sequence and $\{x_n\}$ be a sequence that converges to 0. Show that $\lim y_n x_n = 0$. Find a counter-example showing that we must assume that $\{y_n\}$ is bounded.

Problem 20: Show that the set of irrational numbers is dense in $\mathbb{R}$.

Problem 21: Show that $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \leq \frac{1}{\sqrt{2n+1}}$ and use this to compute $\lim \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$.

Problem 22: Show that if $\lim x_n = a$ and $\lim(x_n - y_n) = 0$, then $\lim y_n = a$.