Introduction to Abstract Mathematics
Spring 2008
Final Exam Review Problems

Problem 1: Let R be a relation from A to B. We define the *inverse* of R to be the relation

$$R^{-1} = \{(b, a) \in B \times A \,|\, (a, b) \in R\}.$$

If S is a relation from B to C then we define the *composition* of S and R to be the relation

$$S \circ R = \{(a, c) \in A \times C \,|\, \text{ there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}.$$

Prove the following.

a. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

b. If T is a relation from C to D then

$$T \circ (R \circ S) = (T \circ R) \circ S.$$

Problem 2: Let $A = \mathbb{N} \times \mathbb{N}$. Define a relation $\sim$ on A by $(a, b) \sim (c, d)$ if and only if $a^b = c^d$.

a. Show that $\sim$ is an equivalence relation on A.

b. Find the equivalence classes of $(16, 1)$ and $(3, 4)$.

c. Find an equivalence class with exactly 4 elements.

d. Find an equivalence class with infinitely many elements.

Problem 3: Let $A = \mathbb{R} \setminus \{1\}$ and let

$$f(x) = \frac{x+1}{x-1}.$$

a. Show that $f : A \to A$.

b. Determine whether or not A is an injection.

c. Determine whether or not A is a surjection.

d. Does f have an inverse? If so, find it. If not, explain.

Problem 4: If A is a set and $f : A \to \mathcal{P}(A)$ is any function, prove that f is not surjective by showing that $\{a \in A \,|\, a \notin f(a)\}$ is not in the range of f. Conclude that $A \not\sim \mathcal{P}(A)$.

Problem 5: Let $\phi : \mathbb{Q} \to \mathbb{Q}$ be an automorphism of $(\mathbb{Q}, +)$. Prove that there is an $r \in \mathbb{Q}$, $r \neq 0$, so that $\phi(x) = rx$ for all $x \in \mathbb{Q}$. [*Hint:* $r = \phi(1)$.]

Problem 6: Find all subgroups of $(\mathbb{Z}, +)$.

Problem 7: Let G be a group. We define the *center of* G to be

$$Z(G) = \{x \in G \mid gx = xg \text{ for every } g \in G\}.$$

In other words, $Z(G)$ consists of the elements of G which commute with everything in G. Show that $Z(G) < G$.

Problem 8: Determine if the following groups are isomorphic. Be sure to justify your answers.

(a) $\mathbb{Z}_6 \oplus \mathbb{Z}_{10}$ and $\mathbb{Z}_{30}$

(b) $\mathbb{Z}_3 \oplus \mathbb{Z}_6$ and $\mathbb{Z}_{18}$

(c) $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_{15}$

Problem 9: For a positive integer n, let $\mathcal{N}(n)$ be the set of subsets of $\mathbb{N}$ having exactly n elements. Show that for all $n \geq 1$, $\mathcal{N}(n)$ is countable.

Problem 10: Show that $\lim(a_n + b_n) = \lim a_n + \lim b_n$.

Problem 11: Let A, B be open, then the set $A + B = \{x + y | \; x \in A, y \in B\}$ is open.

Problem 12: Let $a \neq 0$. If $\lim \frac{x_n}{a} = 1$, then $\lim x_n = a$.