Exercise 1. Define the following terms: *equivalence class, partition, function, injective, surjective, bijective, image, inverse image, inverse function, finite set, group.*

Exercise 2. Study all of the theorems and proofs given in class and on the homework.

Exercise 3. Let $f : X \to Y$ be a function.

- **a.** Prove that f is surjective if and only if there is a function $g : Y \to X$ so that $f \circ g = \mathrm{Id}_Y$.
- **b.** Prove that f is injective if and only if there is a function $h : Y \to X$ so that $h \circ f = \mathrm{Id}_X$.

Exercise 4. Let X and Y be nonempty sets and let $f : X \to Y$ be a function.

- **a.** Let A and B be (nonempty) subsets of Y. Prove that if $A \cap B = \emptyset$ then $f^{-1}(A) \cap f^{-1}(B) = \emptyset$.
- **b.** Prove that
$$X = \bigcup_{b \in Y} f^{-1}(\{b\}).$$
- **c.** Prove that $\{f^{-1}(\{y\}) \mid y \in Y\}$ is a partition of X if and only if f is onto.

Exercise 5. Let $\mathcal{P} = \{[n, n+1) \mid n \in \mathbb{Z}\}$.

- **a.** Prove that $\mathcal{P}$ is a partition of $\mathbb{R}$.
- **b.** For $x \in \mathbb{R}$ let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Prove that
$$x \equiv y \text{ if and only if } \lfloor x \rfloor = \lfloor x \rfloor$$
is an equivalence relation on $\mathbb{R}$.
- **c.** Prove that $\mathcal{P} = \mathbb{R}/\equiv$.

Exercise 6. Let $(G, *)$ be a group. Let $u \in G$ and define $\star : G \times G \to G$ by $a \star b = a * u^{-1} * b$ for all $a, b \in G$. Is $(G, \star)$ a group?

Exercise 7. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Find a bijection between $[a, b]$ and $[c, d]$.

Exercise 8. Let $i = \sqrt{-1} \in \mathbb{C}$ and $C = \{\pm 1, \pm i\}$.

a. Prove that $(C, \cdot)$ is an abelian group.

b. Compute the Cayley tables of $(C, \cdot)$ and $(\mathbb{Z}_4, +_4)$.

c. Find a function $f : \mathbb{Z}_4 \to C$ that, when applied to the Cayley table of $\mathbb{Z}_4$, produces the Cayley table of C.

Exercise 9. If n is a composite natural number, prove that $\cdot_n$ is *not* a binary operation on $\mathbb{Z}_n - \{0\}$.

Exercise 10. Prove that $\mathbb{Q}$ is infinite.

Exercise 11. Prove that $f : (-1, 1) \to \mathbb{R}$, given by $f(x) = x/(x^2 - 1)$, is a bijection and find its inverse.

Exercise 12. Consider the following definitions.

Definition: A set $A \subseteq \mathbb{Z}$ is called *bounded* if there is an $M \in \mathbb{N}$ so that $|n| \leq M$ for all $n \in A$.

Definition: If $A \subseteq \mathbb{Z}$, m is called the *largest element* of A if $m \in A$ and $n \leq m$ for all $n \in A$.

Prove the following result.

Characterization of Finite Subsets of $\mathbb{N}$. Let $\emptyset \neq A \subset \mathbb{N}$. The following are equivalent.

i. A is finite.

ii. A is bounded.

iii. A has a maximum element.

Exercise 13. Let X and Y be nonempty sets and define

$$\mathcal{F}(X, Y) = \{f : X \to Y\}.$$

Fix $a \in X$. Show that $\epsilon_a : \mathcal{F}(X, Y) \to Y$ given by $\epsilon_a(f) = f(a)$ is surjective.

Exercise 14. Let X be a nonempty set and $\xi \in X$. Define $\mathcal{F}_\xi = \{f \in \mathrm{Aut}(X) \mid f(\xi) = \xi\}$. Prove that $(\mathcal{F}_\xi, \circ)$ is a group.

Exercise 15. The picture below shows actor Tom Skerritt in a famous movie from 1979. Name this film, or another well-known film he appeared in.

Exercise 16. Let $(G_1, *_1)$ and $(G_2, *_2)$ be groups. Let $G = G_1 \times G_2$ and for $a_1, a_2 \in G_1$ and $b_1, b_2 \in G_2$ define $(a_1, b_1) * (a_2, b_2) = (a_1 *_1 a_2, b_1 *_2 b_2)$. Prove that $(G, *)$ is a group.