Math 3326 Fall 2009

Introduction to Abstract Mathematics

Final Exam

Due Friday, December 18, 5:00 pm

Your name (please print):

Instructions: This is an untimed, open notes, take home exam. You must complete 10 of the given 11 problems. All problems are worth 10 points each. Unless otherwise indicated, you must justify all of your answers to receive full credit. Be sure to write orderly, legibly and in complete sentences (where appropriate). Please write on only one side of each page, and put each problem on a separate sheet. Notation is important, and points will be deducted for incorrect use. While I expect everyone to work independently, you should feel free to ask me questions.

The Honor Code requires that you do not discuss this exam with any other students or faculty until after the due date.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Points	10	10	10	10	10	10	10	10	10	10
Score										

Total:____________________

1. Definitions.

a. Define what it means for a set G to be a *group*.

b. Let $f : X \to Y$ be a function. Define what it means for f to be an *injection*; a *surjection*; a *bijection*.

c. Let $m, n \in \mathbb{Z}$. Define what it means when we say m *divides* n

d. Let G be a group, $a \in G$. Define the *order* of a.

e. Let S be a set. Define what it means for R to be a *relation* on S. Define what it means for R to be an *equivalence relation* on S.

f. Let $n \in \mathbb{N}$, $n \geq 2$. Define $\mathbb{Z}_n$ and $U(n)$.

g. Let X be a set. Define what it means for X to be *countable*.

2. Prove that for all $n \in \mathbb{N}$, if $A_1, A_2, \ldots, A_n$ are countable then $\displaystyle\bigcup_{i=1}^{n} A_i$ is countable. [*Hint:* Induct on n, with $n = 2$ as the base case.]

3. Let G be a group and let $a \in G$. Prove that $\langle a \rangle$ is a subgroup of G.

4. Let $S = \{r \in \mathbb{R} \,|\, r^2 + ar + b = 0 \text{ for some } a, b \in \mathbb{Q}\}$. Prove that S is countable.

5. Let G be a cyclic group.

a. If G is finite and $|G| = n$, prove that $G \cong \mathbb{Z}_n$.

b. If G is infinite, prove that $G \cong \mathbb{Z}$.

6. Let G be a group and let $\mathrm{Iso}(G) = \{\varphi : G \to G \,|\, \varphi \text{ is an isomorphism}\}$.

a. Prove that $\mathrm{Iso}(G)$ is a subgroup of $\mathrm{Aut}(G)$.

b. We proved in the homework that if $\varphi \in \mathrm{Iso}(\mathbb{Q})$ then $\varphi(x) = ax$ for some $a \in \mathbb{Q}^\times = \mathbb{Q} - \{0\}$. Use this to show that $\mathrm{Iso}(\mathbb{Q}) \cong \mathbb{Q}^\times$. Remember, the operation in $\mathrm{Iso}(\mathbb{Q})$ is composition and the operation in $\mathbb{Q}^\times$ is multiplication.

7. Let $n \in \mathbb{N}$, $n \geq 2$. For $i \in \{1, 2, \ldots, n\}$ let $F_i = \{\sigma \in S_n \,|\, \sigma(i) = i\}$. Prove that F_i is a subgroup of S_n.

8. Let G be a group and $H \leq G$. Given $x, y \in G$, define $x \sim y$ if and only if $xy^{-1} \in H$. Prove that $\sim$ is an equivalence relation on G.

9. Given sets A, B, recall that their *symmetric difference* is $A \Delta B = (A - B) \cup (B - A)$. Also recall that in a previous exercise you proved that for any three sets A, B, C one has

$(A\Delta B)\Delta C = A\Delta(B\Delta C)$. Use this fact to prove that if X is a nonempty set then $(\mathcal{P}(X), \Delta)$ is a group.

10. Let $Q = \{\pm 1, \pm a, \pm b, \pm c\}$ subject to the multiplication rules $(-1)x = x(-1) = -x$ for all $x \in Q$ and $a^2 = b^2 = c^2 = abc = -1$. Construct the Cayley table for Q. Assuming associativity, is Q a group? If so, is Q abelian?

11. Let G be a set with a binary operation. Suppose the binary operation has the following properties:

i. $(ab)c = a(bc)$ for all $a, b, c \in G$.

ii. There is an $e \in G$ so that $ae = ea = a$ for all $a \in G$.

iii. For all $a \in G$ there is a $b \in G$ so that $ab = e$.

Prove that if G is finite then G is a group. [*Hint:* The only question is the existence of inverses. To answer it, given $a \in G$ define $f : G \to G$ by $f(x) = xa$, and show that f is onto.]