Homework #10 Solutions

p 165, #4 Let G and H be groups and let $x, y \in G$, $u, v \in H$. Then $(x,u),(y,v) \in G \oplus H$. By the definition of the binary operation in $G \oplus H$ we have

$$\begin{aligned}(x,u)(y,v) &= (xy,uv)\\ (y,v)(x,u) &= (yx,vu).\end{aligned}$$

Since two elements of $G \oplus H$ are equal if and only if their corresponding coordinates are equal, it is clear from this pair of equations that $G \oplus H$ is abelian if and only if both G and H are abelian.

p 165, #6 Since 1 has order 8 in $\mathbb{Z}_8$, the order of $(1,0) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$ is $\text{lcm}(|1|,|0|) = \text{lcm}(8,1) = 8$. However, given any $(a,b) \in \mathbb{Z}_4 \oplus \mathbb{Z}_4$ we have

$$4(a,b) = (4a \bmod 4, 4b \bmod 4) = (0,0)$$

so that $|(a,b)|$ must divide 4. In particular, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ contains no elements of order 8. From this it follows that $\mathbb{Z}_8 \oplus \mathbb{Z}_2 \not\cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

p 165, #12 Let K be the subgroup of rotations in D_n and let H be any subgroup of D_n with order 2. Then K and H are both cyclic, since $K = \langle R_{360/n} \rangle$ and $H \cong \mathbb{Z}_2$. Since cyclic groups are abelian, exercise #4 implies that $K \oplus H$ is also abelian. Since D_n is *not* abelian, we must have $D_n \not\cong K \oplus H$.

p 166, #14 Let $\phi : G_1 \to G_2$, $\psi : H_1 \to H_2$ be isomorphisms. Define

$$\begin{aligned}\phi \oplus \psi : G_1 \oplus H_1 &\to G_2 \oplus H_2\\ (g,h) &\mapsto (\phi(g), \psi(h)).\end{aligned}$$

We will prove that this is an isomorphism.

If $\phi \oplus \psi(a,b) = \phi \oplus \psi(g,h)$ then, by definition, $(\phi(a),\psi(b)) = (\phi(g),\psi(h))$, and so $\phi(a) = \phi(g)$ and $\psi(b) = \psi(h)$. Since ϕ and ψ are both isomorphisms, they are both one-to-one, so $a = g$ and $b = h$. But this means $(a,b) = (g,h)$, proving that $\phi \oplus \psi$ is one-to-one.

Let $(g,h) \in G_2 \oplus H_2$. Then $g \in G_2$, $h \in H_2$ and since ϕ, ψ are both onto, there exist $a \in G$, $b \in H$ so that $\phi(a) = g$ and $\psi(b) = h$. But then $(a,b) \in G_1 \oplus H_1$ and

$$\phi \oplus \psi(a,b) = (\phi(a),\psi(b)) = (g,h)$$

and $\phi \oplus \psi$ is onto.

Finally, let $(a,b),(g,h) \in G_1 \oplus H_1$. Then we have

$$\begin{aligned} \phi \oplus \psi((a,b)(g,h)) &= \phi \oplus \psi(ag, bh) \\ &= (\phi(ag), \psi(bh)) \\ &= (\phi(a)\phi(g), \psi(b)\psi(h)) \\ &= (\phi(a), \psi(b))(\phi(g), \psi(h)) \\ &= (\phi \oplus \psi(a,b))(\phi \oplus \psi(g,h)) \end{aligned}$$

since both ϕ and ψ preserve operations. Thus $\phi \oplus \psi$ is operation preserving as well.

p 166, #20 Since S_3 is nonabelian, exercise #4 tells us that $S_3 \oplus \mathbb{Z}_2$ is nonabelian as well, and hence we can eliminate the abelian groups $\mathbb{Z}_{12}$ and $\mathbb{Z}_6 \oplus \mathbb{Z}_2$. Notice that since $(123) \in S_3$ has order 3 and $1 \in \mathbb{Z}_2$ has order 2, the element $((123), 1)$ in $S_3 \oplus \mathbb{Z}_2$ has order $\text{lcm}(3,2) = 6$. But S_4 has no elements of order 6, which means that neither does the subgroup A_4. We can therefore eliminate A_4 from our list of candidates and conclude that $S_3 \oplus \mathbb{Z}_2 \cong D_6$.

p 166, #22 Consider $(2,1) \in \mathbb{Z}_4 \oplus \mathbb{Z}_2$. Since $2(2,1) = (4 \bmod 4, 2 \bmod 2) = (0,0)$, this element has order 2. Therefore, $\langle(2,1)\rangle$ is a subgroup of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ of order 2. Let $H \leq \mathbb{Z}_4$ and $K \leq \mathbb{Z}_2$. If $(2,1) \in H \oplus K$ then $2 \in H$ and $1 \in K$. Hence, $\{0,2\} \leq H$ and $K = \mathbb{Z}_2$, which means that $|H \oplus K| = |H| \cdot |K| \geq 2 \cdot 2 = 4$. Since $4 > 2$ it is impossible to have $\langle(2,1)\rangle = H \oplus K$.