Homework #6 Solutions

p 112, # 6. Since the LCM of 3 and 5 is 15, any element of S_8 of the form $\sigma = (a_1a_2a_3)(a_4a_5a_6a_6a_8)$ with the a_i all distinct will have order 15. Since 3 and 5 are odd, any 3-cycle or 5-cycle is an even permutation and therefore belongs to A_8. It follows that every element of S_8 of the form $\sigma = (a_1a_2a_3)(a_4a_5a_6a_6a_8)$ has order 15 and belongs to A_8. One choice for such an element is $\sigma = (123)(45678)$.

p 112, # 8. Recall that the disjoint cycle structures of elements in S_{10} correspond to the partitions of 10,

$$l_1 + l_2 + \cdots l_k = 10 \ , \ l_1 \geq l_2 \geq \cdots \geq l_k \geq 1,$$

and that the order of an element made up of disjoint cycles of lengths $l_1, l_2, \ldots, l_k$ is $\text{lcm}(l_1, l_2, \cdots, l_k)$. We have seen that an l-cycle can be written as the product of $l-1$ transpositions, and it follows that given a partition $l_1 + l_2 + \cdots l_k = 10$ any corresponding permutation can be written as a product of $(l_1 - 1) + (l_2 - 1) + \cdots (l_k - 1) = 10 - k$ transpositions. Hence, such a permutation is even if and only if k is even.

So, to determine the possible orders of elements of A_{10} we need only determine the partitions of 10 into an even number of integers:

Partition	LCM
$1+1+\cdots+1$	1
$2+2+1+\cdots+1$	2
$2+2+2+2+1+1$	2
$3+1+\cdots+1$	3
$3+2+2+1+1+1$	6
$3+3+1+1+1+1$	3
$3+3+2+2$	6
$3+3+3+1$	3
$4+2+1+1+1+1$	4
$4+2+2+2$	2
$4+3+2+1$	12
$4+4+1+1$	4
$5+1+1+1+1+1$	5
$5+2+2+1$	10
$5+3+1+1$	15
$5+5$	5
$6+2+1+1$	6
$6+4$	12
$7+1+1+1$	7
$7+3$	21
$8+2$	8
$9+1$	9

We see immediately that the largest order possible is 21.

p 113, # 14. If α is an r-cycle, β is an s-cycle and γ is a t-cycle, then α can be written as $r-1$ transpositions, β as $s-1$ transpositions, and γ as $t-1$ transpositions. It follows that $\alpha\beta$ can be written as $r+s-2$ transpositions and $\alpha\beta\gamma$ can be written as $r+s+t-3$ transpositions. Hence, $\alpha\beta$ is even if and only if $r+s$ is even and $\alpha\beta\gamma$ is even if and only if $r+s+t$ is odd.

p 113, # 18b. We have, according to the proof of Theorem 5.4,

$$\begin{aligned} \alpha &= (12345)(678) = (15)(14)(13)(12)(68)(67) \\ \beta &= (23847)(56) = (27)(24)(28)(23)(56). \end{aligned}$$

However, these expressions are not unique.

p 113, # 22. Since A_n is a subgroup of S_n, we find that for any $\sigma \in S_n$, σ is even if and only if σ^{-1} is even. So in the product $\alpha^{-1}\beta^{-1}\alpha\beta$, α and α^{-1} together contribute an even number of transpositions and so do β and β^{-1}. It follows immediately that $\alpha^{-1}\beta^{-1}\alpha\beta \in A_n$.

p 114, # 26. (1234) has length 4, so is odd. Since every 3-cycle is even, and therefore belongs to A_n, every product of 3-cycles belongs to A_n. Thus, if (1234) were a product of 3-cycles it would be even, a contradiction.

p 114, # 42. As in #22, the terms β and β^{-1} together contribute an even number of transpositions to the product $\beta\alpha\beta^{-1}$. Hence, the number of transpositions in $\beta\alpha\beta^{-1}$ has the same parity as the number of transpositions in α, i.e. $\beta\alpha\beta^{-1}$ is even precisely when α is even. The conclusion follows.

p 115, # 50. We can view the output of one run of the shuffling machine as an element $\sigma \in S_{13}$, where we label our set of permuted objects with the names of the cards in the deck: $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}$. With this convention, we are given that

$$\sigma^2 = \begin{pmatrix} A & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & J & Q & K \\ 10 & 9 & Q & 8 & K & 3 & 4 & A & 5 & J & 6 & 2 & 7 \end{pmatrix} = (A\,10\,J\,6\,3\,Q\,2\,9\,5\,K\,7\,4\,8).$$

We see that σ^2 has order 13. If $|\sigma = n$ then we know that $13 = |\sigma^2| = n/(2,n)$ so that $n = (2,n)13$. This implies that $n = 13$ or $n = 26$. It is easy to see that the latter can not occur (i.e. S_{13} has no element of order 26), so we must have $13 = n = |\sigma|$. It follows that $\langle\sigma\rangle = \langle\sigma^2\rangle$, which means we can write σ as a power of σ^2. Since $2 \cdot 7 \bmod 13 = 1$ we have

$$\begin{aligned} \sigma = \sigma^{14} = (\sigma^2)^7 &= (A\,9\,10\,5\,J\,K\,6\,7\,3\,4\,Q\,8\,2) \\ &= \begin{pmatrix} A & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & J & Q & K \\ 9 & A & 4 & Q & J & 7 & 3 & 2 & 10 & 5 & K & 8 & 6 \end{pmatrix}. \end{aligned}$$