Automorphisms

Exercise 1. Consider the set of all *continuous* automorphisms of $\mathbb{R}$,

$$\mathrm{Aut}_c(\mathbb{R}) = \{f \in \mathrm{Aut}(\mathbb{R}) \mid f \text{ is continuous}\}.$$

(a) Show that $\mathrm{Aut}_c(\mathbb{R})$ is a subgroup of $\mathrm{Aut}(\mathbb{R})$.

(b) Let $\phi \in \mathrm{Aut}_c(\mathbb{R})$. Show that $\phi(x) = x\phi(1)$ for all $x \in \mathbb{R}$. [*Hint:* Show that the equality holds for all $x \in \mathbb{Q}$ and then use continuity to establish it for general x.]

(c) Use part (b) to show that $\mathrm{Aut}_c(\mathbb{R}) \cong \mathbb{R}^\times$.

Exercise 2. Let $\phi : G \to H$ be an isomorphism between the groups G and H. Show that the function $\hat{\phi} : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ defined by

$$\hat{\phi}(f) = \phi \circ f \circ \phi^{-1}$$

for all $f \in \mathrm{Aut}(G)$ is an isomorphism.