Math 3362 Fall 2006

Modern Algebra

Final Exam Practice Problems

Problem 1. The following is a well-known fact about finite groups.

Proposition 1. *If G is a group and $|G| = p^n$ for some prime p and $n \in \mathbb{Z}^+$, then $|Z(G)| > 1$.*

Use this proposition and the G/Z Theorem to prove that every group of order p^2, p prime, is abelian.

Problem 2. The Fundamental Theorem of Finite Abelian Groups implies that $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \not\cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$. Prove this directly (please don't build a Cayley table).

Problem 3. Let $G = U(32)$ and $H = \{1, 31\}$. Then G/H is abelian of order 8, so is isomorphic to exactly one of $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine which one.

Problem 4. Let G be a group and $N \triangleleft G$. Suppose $N \leq H \leq G$. Then $N \triangleleft H$ and $H/N \leq G/N$ (you don't need to prove this). Show that $H/N \triangleleft G/N$ if and only if $H \triangleleft G$.

Problem 5. Use the first isomorphism theorem and the determinant map to prove that $\mathrm{GL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{R}) \cong \mathbb{R}^{\times}$.

Problem 6. Let G be a group and suppose $N \triangleleft G$ with $[G : N] = 5$. Use this information to construct a nontrivial homomorphism $\phi : G/N \to D_5$.

Problem 7. Show that $\mathrm{Aut}(U(8)) \cong S_3$.

Problem 8. Express $\mathrm{Aut}(U(343))$ as a product of cyclic groups of prime power order.

Problem 9. How many non-isomorphic abelian groups of order 1729 are there?

Problem 10. Let p, q be odd primes and $m, n \in \mathbb{Z}^+$. Show that $U(p^m) \oplus U(q^n)$ is not cyclic.

Problem 11. Let a, b be relatively prime nonzero integers.

(a) Show that the map $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ defined by $(x, y) \mapsto bx - ay$ is an onto homomorphism.

(b) Show that $\ker \phi = \langle (a, b) \rangle$.

(c) Show that $(\mathbb{Z} \oplus \mathbb{Z})/\langle (a, b) \rangle \cong \mathbb{Z}$.

Problem 12. Let $\phi : \mathbb{Z}_{17} \to G$ be a homomorphism. If ϕ is not one-to-one, determine ϕ.

Problem 13. Let $\phi : \mathbb{Z}_{60} \to G$ be an onto homomorphism.

(a) If $|\ker \phi| = 4$, determine G (up to isomorphism).

(b) If $|G| = 10$, determine $\ker \phi$.

Problem 14. Let G, H be groups. Show that the *projection*

$$\begin{aligned} \pi : G \oplus H &\rightarrow G \\ (g, h) &\mapsto g \end{aligned}$$

is a homomorphism. Determine $\ker \pi$ and $\mathrm{Im}\pi$.

Problem 15. Let $\phi : G \rightarrow H$ be a homomorphism of groups. Prove that for any $x, y \in G$, $\phi(x) = \phi(y)$ if and only if $x(\ker \phi) = y(\ker \phi)$.

Problem 16. How many homomorphisms are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_{10}$?

Problem 17. Let $\phi : G \rightarrow \mathbb{Z}_6 \oplus \mathbb{Z}_3$ be a homomorphism with $|\ker \phi| = 5$. Prove that G must have normal subgroups of sizes 5, 10, 15, 20, 30 and 60.

Problem 18. Prove that $(\mathbb{Z}_{17} \oplus \mathbb{Z}_{26})/\langle(8, 5)\rangle$ is cyclic. What is its order?

Problem 19. Prove or disprove:

$$\mathbb{Z}_{100} \oplus \mathbb{Z}_{50} \oplus \mathbb{Z}_{25} \cong \mathbb{Z}_{25} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_8.$$

Problem 20. State and prove the First Isomorphism Theorem.