Problem 1. The following is a well-known fact about finite groups.

**Proposition 1.** If $G$ is a group and $|G| = p^n$ for some prime $p$ and $n \in \mathbb{Z}^+$, then $|Z(G)| > 1$.

Use this proposition and the $G/Z$ Theorem to prove that every group of order $p^2$, $p$ prime, is abelian.

Problem 2. The Fundamental Theorem of Finite Abelian Groups implies that $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \not\cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$. Prove this directly (please don’t build a Cayley table).

Problem 3. Let $G = U(32)$ and $H = \{1, 31\}$. Then $G/H$ is abelian of order 8, so is isomorphic to exactly one of $\mathbb{Z}_8$, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine which one.

Problem 4. Let $G$ be a group and $N \triangleleft G$ with $[G : N] = 5$. Use this information to construct a nontrivial homomorphism $\phi : G/N \rightarrow D_5$.

Problem 5. Use the first isomorphism theorem and the determinant map to prove that $GL(2, \mathbb{R})/SL(2, \mathbb{R}) \cong \mathbb{R}^\times$.

Problem 6. Let $G$ be a group and suppose $N \triangleleft G$ with $[G : N] = 5$. Use this information to construct a nontrivial homomorphism $\phi : G/N \rightarrow D_5$.

Problem 7. Show that $\text{Aut}(U(8)) \cong S_3$.

Problem 8. Express $\text{Aut}(U(343))$ as a product of cyclic groups of prime power order.

Problem 9. How many non-isomorphic abelian groups of order 1729 are there?

Problem 10. Let $p, q$ be odd primes and $m, n \in \mathbb{Z}^+$. Show that $U(p^m) \oplus U(q^n)$ is not cyclic.

Problem 11. Let $a, b$ be relatively prime nonzero integers.

(a) Show that the map $\phi : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $(x, y) \mapsto bx - ay$ is an onto homomorphism.

(b) Show that $\ker \phi = \langle (a, b) \rangle$.

(c) Show that $(\mathbb{Z} \oplus \mathbb{Z})/\langle (a, b) \rangle \cong \mathbb{Z}$.

Problem 12. Let $\phi : \mathbb{Z}_{17} \rightarrow G$ be a homomorphism. If $\phi$ is not one-to-one, determine $\phi$.

Problem 13. Let $\phi : \mathbb{Z}_{60} \rightarrow G$ be an onto homomorphism.

(a) If $|\ker \phi| = 4$, determine $G$ (up to isomorphism).

(b) If $|G| = 10$, determine $\ker \phi$. 

Problem 14. Let $G, H$ be groups. Show that the projection
\[ \pi : G \oplus H \to G \]
\[ (g, h) \mapsto g \]
is a homomorphism. Determine $\ker \pi$ and $\text{Im} \pi$.

Problem 15. Let $\phi : G \to H$ be a homomorphism of groups. Prove that for any $x, y \in G$, $\phi(x) = \phi(y)$ if and only if $x(\ker \phi) = y(\ker \phi)$.

Problem 16. How many homomorphisms are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_{10}$?

Problem 17. Let $\phi : G \to \mathbb{Z}_6 \oplus \mathbb{Z}_3$ be a homomorphism with $|\ker \phi| = 5$. Prove that $G$ must have normal subgroups of sizes 5, 10, 15, 20, 30 and 60.

Problem 18. Prove that $(\mathbb{Z}_{17} \oplus \mathbb{Z}_{26})/\langle (8, 5) \rangle$ is cyclic. What is its order?

Problem 19. Prove or disprove:
\[ \mathbb{Z}_{100} \oplus \mathbb{Z}_{50} \oplus \mathbb{Z}_{25} \cong \mathbb{Z}_{25} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_8. \]

Problem 20. State and prove the First Isomorphism Theorem.