Modern Algebra 1 Exam 2 Review
Spring 2010

Exercise 1. State and prove the following theorems.

a. Cayley's Theorem

b. Lagrange's Theorem

c. The classification of cyclic groups

d. The classification of normal subgroups

e. The First Isomorphism Theorem

Exercise 2. Let R_1 and R_2 be rings.

a. Since $(R_1, +)$ and $(R_2, +)$ are abelian groups, we know that $R_1 \times R_2$ is an abelian group under component-wise addition. Show that $R_1 \times R_2$ is a ring if we define multiplication component-wise as well.

b. Show that if R_1 and R_2 both have unity then so does $R_1 \times R_2$. Prove an analogous statement for commutativity.

c. Show that if R_1 and R_2 both have unity then $(R_1 \times R_2)^\times = R_1^\times \times R_2^\times$.

Exercise 3.[Aut($\mathbb{Z}_n$) and $U(n)$, part 2] We now know that $U(n) = \{m \in \mathbb{Z}_n \mid \gcd(m, n) = 1\}$ is a group under multiplication mod n. We also have seen that given any $f \in \text{Aut}(\mathbb{Z}_n)$ there is a $k \in U(n)$ so that $f(1) = k$. Use this to prove that Aut($\mathbb{Z}_n$) (which is a group under composition) is isomorphic to $U(n)$.

Exercise 4. Let R be a ring. An element $x \in R$ is called *nilpotent* if there is an $n \in \mathbb{N}$ so that $x^n = 0$.

a. If R is commutative and $x, y, z \in R$ with y and z nilpotent, prove that $y - z$ and xy are both nilpotent.

b. If R is a commutative ring, show that $M_2(R)$ always has nonzero nilpotent elements.

Exercise 5.[Proof of the Correspondence Principle] Let G be a group and $N \lhd G$. Let $S_1 = \{H \,|\, N \leq H \leq G\}$ and $S_2 = \{K \,|\, K \leq G/N\}$. Let $F : S_1 \to S_2$ be given by $F(H) = H/N$.

a. Prove that F is a bijection. [*Suggestion:* Find F^{-1}.]

b. Let $H \in S_1$. Prove that $H \lhd G$ if and only if $F(H) \lhd G/N$.

c. Let $H_1, H_2 \in S_1$. Prove that $H_1 \leq H_2$ if and only if $F(H_1) \leq F(H_2)$.

Exercise 6. Let G be the set of all rational functions on $\mathbb{R}$ of the form $f(x) = \dfrac{ax+b}{cx+d}$ where $ad - bc \neq 0$ (such a function is called a *fractional linear transformation*).

a. Prove that G is a group under function composition.

b. Define $\varphi : \mathrm{GL}_2(\mathbb{R}) \to G$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \dfrac{ax+b}{cx+d}$. Prove that φ is a homomorphism and compute its kernel.

Exercise 7. Let G be a group and $H \leq G$. Recall the homomorphism $T : G \to \mathrm{Sym}(G/H)$ given by $T(x) = T_x$, where $T_x(yH) = xyH$ for all $y \in G$. Prove that the kernel of T is the *normal core* of H, $\mathrm{Core}(H) = \bigcap_{x \in G} xHx^{-1}$.

Exercise 8. A group G is called *solvable* is there is a sequence of subgroups

$$\{e\} = H_0 \leq H_1 \leq \cdots \leq H_{r-1} \leq H_r = G$$

so that $H_i \lhd H_{i+1}$ for all i and H_{i+1}/H_i is abelian for all i. Prove that if G is solvable and $K \leq G$ then K is solvable. Also prove that if $K \lhd G$ then G/K is solvable, too.