Math 3362 Spring 2010

Modern Algebra I

Final Exam

Due: Tuesday, May 11, 5:00 pm

Your name (please print):

Instructions: This is an untimed open book, open notes exam. You may freely consult your lecture notes, homework and course textbook (indeed, you are fully expected to), but no other resources are permitted. You must carefully cite any results that you choose to quote (e.g. "By the First Isomorphism Theorem…" or "By Theorem 2.18…" or "We proved in class that…"). Poorly written, sloppy, or unjustified answers will receive partial credit at best. Be sure to staple this page to the front of your exam solutions when you turn them in.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Points	10	10	10	10	10	10	10	10	10	10
Score										

Total:____________________

Computations

1. Find the cycle decompositions, order, and parity (even/odd-ness) of each of the following permutations.

a. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 5 & 7 & 1 & 6 & 10 & 3 & 2 & 8 & 4 \end{pmatrix}$ **b.** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 9 & 4 & 6 & 7 & 5 & 10 & 8 & 3 \end{pmatrix}$

c. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 6 & 5 & 7 & 9 & 1 & 8 & 2 & 10 & 4 \end{pmatrix}$ **d.** $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 9 & 10 & 3 & 7 & 1 & 5 & 6 & 2 & 8 \end{pmatrix}$

2. Find representatives for the conjugacy classes in S_4 and determine the order of each class.

3. Write your birthday in the standard format $MM - DD - YYYY$ (for example, if you were born on May 5, 2010, you'd write $5 - 05 - 2010$). Use these numbers to form a 7 to 8 digit number n (so in our example, we'd get $n = 5052010$). Add this number to your student ID number and call the sum N. Write down a list of abelian groups so that any abelian group of order N is isomorphic to exactly one group in your list.

4. Let G be a finite group, let p be a prime number, and suppose that p divides $|G|$. Let N_p denote the number of elements of order p in G. In Problem 7 below you will show that N_p is divisible by $\varphi(p) = |U(p)|$. One can also show that $N_p + 1$ is always divisible by p (in particular, $N_p \neq 0$). Use these facts to prove that groups of order 15, 35 or 77 are always cyclic.

Proofs

5. Let $n \in \mathbb{N}$, $n \geq 2$, and let $H \leq S_n$. Prove that $[H : H \cap A_n] = 1$ or 2. Conclude that either every element of H is even or that exactly half of them are.

6. Let G be a finite group and suppose that the only automorphism of G is the identity. Prove that G is isomorphic to $\mathbb{Z}_2$.

7. Let G be a finite group, m a natural number, and suppose that m divides $|G|$. If $\varphi(m) = |U(m)|$ and N_m denotes the number of elements in G with order m, prove that N_m is divisible by $\varphi(m)$.

8. Let G be a finite group, let $N \lhd G$, and let $H \leq G$.

a. If $|H|$ and $|N|$ are relatively prime prove that $|HN| = |H| \cdot |N|$.

b. If $|H|$ equals $[G : N]$ and is relatively prime to $|N|$ prove that $G = HN$.

9. Let R be a ring (with unity). Recall that an element $r \in R$ is called *nilpotent* if there is an $n \in N$ so that $r^n = 0$. If $r \in R$ is nilpotent, prove that $1 + r \in R^\times$. Does the converse of this statement hold?

10. Let p be a prime number and let

$$\begin{aligned} \mathbb{Z}_{(p)} &= \{a/b \in \mathbb{Q} \,|\, p \nmid b\}, \\ \mathbb{Z}[1/p] &= \{a/b \in \mathbb{Q} \,|\, b = p^k \text{ for some } k \in \mathbb{Z}, k \geq 0\}. \end{aligned}$$

Prove that $\mathbb{Z}_{(p)}$ and $\mathbb{Z}[1/p]$ are rings under ordinary addition and multiplication of rational numbers. Determine the unit groups in both of these rings.

Hints & Suggestions

2. Remember that conjugation is a group action and that we proved that the order of the orbit of an element under a group action is equal to the index of its stabilizer.

4. If G has order pq, where p and q are distinct primes, one way to show G is cyclic is to show that $N_p + N_q + 1 \neq pq$. Why?

5. There are several options here. One can either use the second isomorphism theorem or appeal to the sign homomorphism $\epsilon : S_n \to \{\pm 1\}$.

6. First show that G is abelian and that all its elements have order dividing 2. Then appeal to the appropriate "fundamental theorem."

7. If $N_m \neq 0$, given $a, b \in G$ of order m, define $a \sim b$ if and only if $\langle a \rangle = \langle b \rangle$. Prove that $\sim$ is an equivalence relation and that each equivalence class has size $\varphi(m)$.

8. For part **a**, show that distinct elements of H lie in distinct cosets of N.

9. Show that the formal geometric series identity $(1 + r)^{-1} = 1 - r + r^2 - r^3 + \cdots$ can be interpreted literally in R.