Modern Algebra 1 — Homework 3.2
Spring 2010 — Due February 3

Exercise 6. Given a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, recall that it's *determinant* is $\det A = ad - bc$. The *transpose* of A is the matrix $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Prove the following properties of the determinant and the transpose. You can assume that the matrices involved all have real entires.

- **a.** If A and B are both 2×2 matrices then $\det AB = \det A \det B$.
- **b.** If A and B are both 2×2 matrices, then $(AB)^T = B^T A^T$.
- **c.** If $\det A \neq 0$ (so that A^{-1} exists) then $(A^T)^{-1} = (A^{-1})^T$.

Exercise 7. Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z} \text{ and } ad - bc \neq 0 \right\}$. Prove or disprove: G is a group under matrix multiplication.

Exercise 8. Prove that $\mathrm{O}_2(\mathbb{R}) = \{A \in \mathrm{M}_2(\mathbb{R}) \,|\, AA^T = I\}$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. $\mathrm{O}_2(\mathbb{R})$ is called the *orthogonal group* and its elements are the *orthogonal matrices*.

Exercise 9. Let G be a group, $H \leq G$ and $x \in G$. Prove that $xHx^{-1} = \{xyx^{-1} \,|\, y \in H\}$ is a subgroup of G.