Modern Algebra 1
Spring 2010

Homework 3.3
Due February 3

Exercise 10. The set $\mathrm{GL}_2(\mathbb{C})$ of invertible 2×2 matrices with complex entries can be shown to be a group under matrix multiplication. In fact, the proof given for matrices with real entries can be used, *mutatis mutandis*, do deal with the case of complex entries. Consider the matrices $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Find all the elements in the subgroup $Q = \langle A, B \rangle$ of $\mathrm{GL}_2(\mathbb{C})$. [*Hint:* Show that A and B both have finite order and that $BA = A^3B$. Use this to prove that every element in Q can be written in the form A^iB^j, with only a limited number of possibilities for i and j.]

Exercise 11. Let $m, n \in \mathbb{Z}$, $H = \langle m \rangle$ and $K = \langle n \rangle$. Find (with proof) a necessary and sufficient condition for us to have $H \leq K$.

Exercise 12. A subgroup H of a group G is called *maximal* if $H \neq G$ and whenever we have $H \leq K \leq G$ then $K = H$ or $K = G$ (i.e. there are no subgroups "between" H and G). Use the result of the previous exercise to determine all the maximal subgroups of $\mathbb{Z}$.

Exercise 13. Let H_i, $i \in I$, be a collection of subgroups of a group G. Prove that $K = \bigcap_{i \in I} H_i$ is a subgroup of G. Must the union of subgroups also be a subgroup?