Modern Algebra 1
Spring 2010

Homework 5.1
Due February 17

Exercise 1. Let $f : G \to H$ be an isomorphism of groups.

- **a.** Let $a \in G$. Prove that $|a| = |f(a)|$.
- **b.** Prove that G is abelian if and only if H is abelian.

Exercise 2. Let $f : G \to H$ be an isomorphism of groups. Prove that the mapping $K \mapsto f(K)$ gives a bijection between the sets of subgroups of G and H, and that $K \leq J \leq G$ if and only if $f(K) \leq f(J) \leq H$.

Exercise 3. Let G be a group and let $a, b \in G$. The *commutator* of a and b is the group element $[a, b] = aba^{-1}b^{-1}$. Let $f : G \to H$ be a group homomorphism. Prove that $\operatorname{Im} f$ is abelian if and only if $\ker f$ contains every commutator in G.

Exercise 4.[$\operatorname{Aut}(\mathbb{Z}_n)$ and $U(n)$ - Part I] Let $n \in \mathbb{N}$, $n \geq 2$. Define $U(n) = \{k \in \mathbb{Z}_n \mid \gcd(n, k) = 1\}$. Let $f \in \operatorname{Aut}(\mathbb{Z}_n)$.

- **a.** Show that $\mathbb{Z}_n = \langle k \rangle$ if and only if $\gcd(n, k) = 1$. [*Hint:* Such a k must have $|k| = n$, and we proved a theorem giving the order of any element in $\mathbb{Z}_n$.]
- **b.** Prove that for all $k \in \mathbb{Z}_n$ we have $f(k) = kf(1)$. Conclude that $\mathbb{Z}_n = \langle f(1) \rangle$.
- **c.** Conclude that $f(1) \in U(n)$.