Exercise 1. Let \( G \) be a group and let \( H, K \triangleleft G \). Prove that if \( K \leq H \) and \( G/K \) is cyclic, then \( G/H \) is cyclic. [Suggestion: Use the Third Isomorphism Theorem and the fact (proven in earlier homework) that a quotient of a cyclic group is cyclic.]

Exercise 2. Let \( R \) be a commutative ring (with unity) and let

\[ M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R \right\}. \]

a. Show that if \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)^\times \) then \( \det(A) = ad - bc \in R^\times \). [Hint: The result of exercise 3.2.6.a is valid for any commutative ring.]

b. Show that if \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R) \) and \( \det(A) = ad - bc \in R^\times \), then \( A \in M_2(R)^\times \).

[Hint: Show that the usual formula yields the inverse of \( A \).]

c. Conclude that \( A \in M_2(R)^\times \) if and only if \( \det(A) \in R^\times \).

Exercise 3. Given that \( \mathbb{Z}_n^\times = U(n) \) for \( n \geq 2 \), prove that \( \mathbb{Z}_n^\times = \mathbb{Z}_n - \{0\} \) if and only if \( n \) is prime. This completes the proof that \( \mathbb{Z}_n \) is a field if and only if \( n \) is prime.

Exercise 4. Let \( A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} \). Determine if \( A \) is a unit in \( M_2(R) \) for the following choices of \( R \).

a. \( R = \mathbb{Q} \)

b. \( R = \mathbb{Z} \)

c. \( R = \mathbb{Z}_n \) (your answer will depend on \( n \))