Exercise 1. Let G be a group and let $H, K \lhd G$. Prove that if $K \leq H$ and G/K is cyclic, then G/H is cyclic. [*Suggestion:* Use the Third Isomorphism Theorem and the fact (proven in earlier homework) that a quotient of a cyclic group is cyclic.]

Exercise 2. Let R be a commutative ring (with unity) and let

$$M_2(R) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \ a, b, c, d \in R \right\}.$$

a. Show that if $A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(R)^\times$ then $\det(A) = ad - bc \in R^\times$. [*Hint:* The result of exercise 3.2.6.a is valid for *any* commutative ring.]

b. Show that if $A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(R)$ and $\det(A) = ad - bc \in R^\times$, then $A \in M_2(R)^\times$. [*Hint:* Show that the usual formula yields the inverse of A.]

c. Conclude that $A \in M_2(R)^\times$ if and only if $\det(A) \in R^\times$.

Exercise 3. Given that $\mathbb{Z}_n^\times = U(n)$ for $n \geq 2$, prove that $\mathbb{Z}_n^\times = \mathbb{Z}_n - \{0\}$ if and only if n is prime. This completes the proof that $\mathbb{Z}_n$ is a field if and only if n is prime.

Exercise 4. Let $A = \left(\begin{array}{cc} 1 & 3 \\ -2 & 4 \end{array} \right)$. Determine if A is a unit in $M_2(R)$ for the following choices of R.

a. $R = \mathbb{Q}$

b. $R = \mathbb{Z}$

c. $R = \mathbb{Z}_n$ (your answer will depend on n)