On the Solutions of Autonomous First Order ODEs

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The goal of this note is to prove some results on the stability of solutions to differential equations of the form

$$\frac{dy}{dt} = f(y).$$

As is discussed in many textbooks, the *equilibrium solutions* $y = k$ (where $f(k) = 0$) often dictate the long term behavior of the general solutions. For example, the asymptotic behavior of the (non-constant) solutions to the initial value problem

$$\frac{dy}{dt} = y(1-y)\,, \qquad y(0) = y_0 \tag{1}$$

is completely determined by how y_0 is related to the equilibria $y = 0$ and $y = 1$. Specifically, one can show that if $y_0 < 0$ then $y \to -\infty$ as t increases, whereas is $y_0 > 0$ then $y \to 1$ as $t \to \infty$. These facts can easily be verified by directly computing the solutions to (1) and are also suggested by it slope field. In fact, textbook authors often simply cite the general appearance of the slope field of an autonomous equation as a verification for the asymptotics of its solutions without actually verifying this rigorously. This is entirely understandable since direct solution of autonomous equations, while frequently possible, is often rather tedious. We intend to address this situation by proving a few general theorems that can be used, in many instances, to carefully justify one's intuition.

We begin with the following result, which provides an existence and uniqueness result for solutions to autonomous equations whose initial values are not at equilibrium.

Proposition 1. *Let $f(y)$ be continuous and positive on an open interval $I = (a, b)$. Let $y_0 \in I$ and consider the initial value problem*

$$\frac{dy}{dt} = f(y)\,, \qquad y(0) = y_0. \tag{2}$$

If

$$A = \int_{y_0}^{a} \frac{1}{f(y)}\,dy \ \textit{ and } \ B = \int_{y_0}^{b} \frac{1}{f(y)}\,dy,$$

then there exists a unique solution to (2) *defined on the interval* (A, B)*, and this solution satisfies* $\lim\limits_{t\to B^-} y(t) = b$ *and* $\lim\limits_{t\to A^+} y(t) = a$.

Proof. The properties of $f(y)$ implies that $1/f(y)$ is positive and continuous on I. We can therefore define

$$F(y) = \int_{y_0}^{y} \frac{1}{f(u)}\,du$$

for $y \in I$, thus obtaining a strictly increasing antiderivative of $1/f(y)$ that satisfies $F(y_0) = 0$. The range of $F(y)$ is precisely (A, B) and, due to its monotonicity, F has a differentiable inverse on (A, B). Define $\phi(t) = F^{-1}(t)$ for $t \in (A, B)$. Then $\phi(0) = F^{-1}(0) = y_0$ and we can implicitly differentiate the relationship $t = F(\phi(t))$ to obtain $1 = F'(\phi(t))\phi'(t) = \phi'(t)/f(\phi(t))$. It follows immediately that $\phi(t)$ is a solution to the initial value problem (2), and the monoticity of F can be used to show that the limit conditions holds as well.

It only remains to establish the uniqueness of $\phi(t)$. Suppose that $\psi(t)$ is another solution to (2) that is defined throughout (A, B). Then the chain rule implies that

$$\frac{d}{dt}F(\psi(t)) = F'(\psi(t))\psi'(t) = \frac{\psi'(t)}{f(\psi(t))} = 1$$

which means that we have $F(\psi(t)) = t + C$, for $t \in (A, B)$ and some constant C. The fact that $\psi(0) = y_0$ implies that $C = 0$, and hence $\psi(t) = F^{-1}(t) = \phi(t)$.

$\square$

Corollary 1. *Let $f(y)$ be continuous and negative on an open interval $I = (a, b)$. Let $y_0 \in I$ and consider the initial value problem* (2). *If*

$$A = \int_{y_0}^{b} \frac{1}{f(y)}\, dy \ \text{ and } \ B = \int_{y_0}^{a} \frac{1}{f(y)}\, dy.$$

then there exists a unique solution to (2) *defined on the interval (A, B), and this solution satisfies $\lim_{t\to A+} y(t) = b$ and $\lim_{t\to B^-} y(t) = a$.*

Proof. Apply the proposition to the function $g(y) = -f(-y)$ on the interval $(-b, -a)$, replacing y_0 with $-y_0$. $\square$

With these facts in hand we can now establish the next result.

Theorem 1. *Suppose that $f(y)$ is differentiable on an interval $I = (a, b)$ and that $f'(y) < 0$ for all $y \in (a, b)$. If $c \in I$ and $f(c) = 0$ then for every $y_0 \in (a, b)$ the initial value problem* (2) *has a unique solution defined for all positive t, and that solution approaches c as $t \to \infty$.*

Proof. Without loss of generality we can assume that $y_0 < c$. Since f' is negative throughout (a, b), f is strictly decreasing there, and since $f(c) = 0$ it must be that f is positive on (a, c). Also, since $f'(c) < 0$ there is an $\epsilon > 0$ so that

$$-\epsilon < \frac{f(c) - f(y)}{c - y} = -\frac{f(y)}{c - y}$$

as y approaches c from below. From this we obtain the bound

$$\frac{1}{\epsilon(c - y)} < \frac{1}{f(y)}$$

which is enough to show that

$$\int_{y_0}^{c} \frac{dy}{f(y)} = \infty.$$

The result now follows from Proposition 1. $\square$

Another useful result is the following, which is naturally suggested by looking at slope fields.

Theorem 2. *Suppose that $f(y)$ is positive and continuous on an interval of the form (a, ∞). Then for any $y_0 > a$ the initial value problem* (2) *has a unique solution, and this solution tends to infinity as t increases. If we replace "positive," (a, ∞) and "infinity" and with "negative," $(-\infty, b)$ and "minus infinity," respectively, the result holds as well.*

Proof. In the first case we can apply Proposition 1 with $b = \infty$. In the second we can use the corollary with $a = -\infty$. $\square$

Although we didn't include it in the statement of the theorem, it follows from the proof that we can determine whether or not our solutions blow up in finite time by simply checking whether or not the integrals defining B or A converge. If $|f(y)|$ increases sufficiently rapidly that the integral converges, then the solution diverges to infinity rather rapidly as well (i.e. in finite time). This isn't really that surprising, but it is not totally apparent from the slope field itself.

Example. Consider the autonomous initial value problem

$$\frac{dy}{dt} = y(1 - y)(y - 2)^2,\ y(0) = y_0.$$

The integration required to solve this problem directly is tedious, and leads to an equation in y and t which cannot be easily solved for y. If we let $f(y) = y(1-y)(y-2)^2$ then, in the notation of the proof of Proposition 1, what we are really saying is that the function $F(y)$ is difficult to compute and its inverse function $F^{-1}(y)$ is even worse. Nonetheless, we can completely describe the behavior and domains of the solutions.

Since $f(y)$ is positive and continuous on $(0, 1)$, we can appeal to Proposition 1. It tells us that for any $y_0 \in (0, 1)$ the initial value problem has a unique solution defined on $(-\infty, \infty)$ (since the integrals defining A and B both diverge) and that these solutions tend to 1 as $t \to \infty$, and 0 as $t \to -\infty$. Similar reasoning using Corollary 1 shows that for $y_0 \in (1, 2)$, where $f(y) < 0$, the initial value problem also has a unique solution defined for all time t that tends to 2 and 1 as t tends to $-\infty$ and ∞, respectively.

For $y_0 \in (-\infty, 0)$, where again $f(y) < 0$, Corollary 1 implies the existence of unique solutions defined on intervals of the form $(-\infty, B)$, where $B > 0$ is finite. These solutions tend to $-\infty$ as $t \to B$, and tend to zero as $t \to -\infty$. That is, we are in the situation discussed above, in which the solution diverges in finite time. We have the reverse of this behavior for $y_0 \in (2, \infty)$, in which the solutions all tend to 2 as $t \to \infty$, but tends to infinity in finite time as t decreases.

From our analysis we can conclude that $y = 0$ is an unstable equilibrium, that $y = 1$ is a stable equilibrium, and $y = 2$ is semistable. Moreover, all solutions with initial values between 0 and 2 tend to the equilibrium solution $y = 1$, all those with initial value greater that 2 tend to $y = 2$, and all of those with negative initial value diverge in finite time.

Example. Consider the autonomous initial value problem

$$\frac{dy}{dt} = \frac{1}{y},\ y(0) = y_0,$$

which has no equilibria. Proposition 1 tells us that for $y_0 \in (0, \infty)$ the solutions to the initial value problem are defined on intervals of the form (A, ∞), with $A < 0$, and tend to zero and infinity, respectively, at the left and right ends of their domains. Corollary 1 gives similar behavior for $y_0 \in (-\infty, 0)$: the solutions exist on intervals of the same form, but tend to minus infinity as $t \to \infty$. Not surprisingly, this description agrees with the behavior of the explicit form of the solution: $y = \pm\sqrt{2t + y_0^2}$.