Differential Equations
Spring 2011

Assignment 5.2
Due February 16

Exercise 1. If $L[y] = y'' + py' + qy$, where p and q are any two functions, show that $L[fg] = fL[g] + gL[f] - qfg + 2f'g'$.

Exercise 2. If $L[y] = y'' + ay' + by$, where a and b are constants, show that

$$L[t^n e^{kt}] = \left(t^n c(k) + nt^{n-1}c'(k) + n(n-1)t^{n-2}\right) e^{kt},$$

where $c(x) = x^2 + ax + b$.

Exercise 3. Let L be as in Exercise 2. Let V_n be the vector space of functions spanned by $\{e^{kt}, te^{kt}, t^2e^{kt}, \ldots, t^ne^{kt}\}$, i.e. the space of all functions of the form $f(t)e^{kt}$ where f is a polynomial of degree at most n. Show that L is a linear transformation from V_n to itself, and that L is invertible if and only if $c(k) \neq 0$.

Exercise 4. Let everything be as above. Show that $T : V_n \to V_{n+1}$, given by $T(y) = ty$, is a linear transformation. If $W_n = \operatorname{Im} T$ and $c(k) = 0$, show that L is an invertible linear transformation from W_n to V_n.

Remark. The previous two exercises provide an algorithm for finding a particular solution to any differential equation of the form $y'' + ay' + by = f(t)e^{kt}$, where f is a polynomial. This is the *method of undetermined coefficients.*

Exercise 5. Let $L[y] = y'' + y' + y$. If $k = 1$, find the matrix M for $L : V_3 \to V_3$ relative to the basis given in Exercise 2. Compute M^{-1} and use this to (easily!) solve the following differential equations.

a. $y'' + y' + y = (t^3 + t + 1)e^t$

b. $y'' + y' + y = (3t^2 + 2t - 1)e^t$

c. $y'' + y' + y = (2t^3 - 3t)e^t$