Math 4342 Fall 2010

Number Theory II

First Midterm Exam

Due Thursday, October 28, 5:00 pm

Your name (please print):

Instructions: This is an untimed take home exam. You may freely consult your lecture notes, homework and course textbook (indeed, you are fully expected to), but no other resources are permitted. Be sure to staple this page to the front of your exam solutions.

You must justify all of your answers to receive credit, and you must carefully cite any results that you choose to quote (e.g. "By Theorem 2.18…" or "We proved in class that…"). Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**
Points	10	10	10	10	10	10
Score						

Total:____________________

1. Prove that $\sum_{m|n} d(m)^3 = \left(\sum_{m|n} d(m)\right)^2$.

2.

a. Let f and g be multiplicative arithmetic functions and let $a \in \mathbb{N}$. Prove that the function h defined by

$$h(n) = \sum_{d^a|n} f\left(\frac{n}{d^a}\right)$$

is also multiplicative. [*Suggestion:* Show that the arithmetic function q that is 1 on the a^{th} powers, and zero elsewhere, is multiplicative.]

b. Prove that Liouville's function satisfies

$$\lambda(n) = \sum_{d^2|n} \mu\left(\frac{n}{d^2}\right).$$

3. Prove or disprove the following statements

a. If f is a multiplicative arithmetic function, then so is

$$F(n) = \prod_{d|n} f(d).$$

b. If f and g are differentiable and $f(x) = O(g(x))$ then $f'(x) = O(\max\{1, |g'(x)|\})$.

4. Given a natural number k, show that there is a constant A_k so that

$$\sum_{\substack{n \le x \\ (n,k)=1}} \frac{1}{n} = \frac{\varphi(k)}{k} \log x + A_k + O\left(\frac{1}{x}\right).$$

5. Prove the following asymptotic formulas.

a. $\sum_{n \le x} \log^2 n = x \log^2 x + O(x \log x)$

b. $\sum_{p \le x} \log^2 p = O(x \log x)$

6. Bertrand's postulate states that for any $x \ge 1$ there is a prime $p \in (x, 2x]$.[1] Show that the Prime Number Theorem implies the following weaker form of Bertrand's postulate: there is an $x_0 > 0$ so that for all $x \ge x_0$ there is a prime $p \in (x, 2x]$. [*Suggestion:* Consider the quantity $\pi(2x) - \pi(x)$.]

[1] This was first proven by Chebyshev in 1852.