MATH 4342 FALL 2010

NUMBER THEORY II

FINAL EXAM

DUE THURSDAY, DECEMBER 16, 5:00 PM

YOUR NAME (PLEASE PRINT):

Instructions: This is an untimed take home exam. You may freely consult your lecture notes, homework and course textbook (indeed, you are fully expected to), but no other resources are permitted. Be sure to staple this page to the front of your exam solutions.

You must justify all of your answers to receive credit, and you must carefully cite any results that you choose to quote (e.g. "By Theorem 2.18..." or "We proved in class that..."). Unjustified answers and/or disorganized or otherwise illegible work will receive partial credit at best. Notation is important, and points will be deducted for incorrect use.

The Honor Code requires that you neither give nor receive any aid on this exam.

Please indicate that you have read and understood these guidelines by signing your name in the space provided:

Pledged: ______________________________

Do not write below this line

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Points	10	10	10	10	10	10	10	10	10	10
Score										

Total:____________________

1. Given an natural number $n \geq 2$, define its *squarefree kernel* to be

$$k(n) = \prod_{p|n} p.$$

Set $k(1) = 1$. Show that $k = u * (\mu^2\varphi)$.

2. Prove that

$$\sum_{p \leq x} \frac{1}{p^2} = B + O\left(\frac{\log\log x}{x}\right)$$

where

$$B = \sum_{p} \frac{1}{p^2} = 2\int_2^\infty \frac{\pi(t)}{t^3}\,dt.$$

3. Prove that

$$\sum_{\substack{n \leq x \\ (n,k)=1}} \frac{\log n}{n} = \frac{\varphi(k)}{2k}\log^2 x + O(1).$$

4. If $\omega(n)$ is the number of distinct primes dividing n, show that there is a constant A so that

$$\frac{1}{x}\sum_{n \leq x} \omega(n) = \log\log x + A + O\left(\frac{1}{\log x}\right).$$

5. Use Bertrand's postulate (that there's always a prime in $[x, 2x)$) to prove that the sum

$$\sum_{m=1}^{n} \frac{1}{m}$$

is not an integer if $n > 1$.

6. Let G be a finite abelian group, and let $\widehat{G}$ denote the character group of G. If H is a subgroup of G, let

$$H^\sharp = \{f \in \widehat{G} \,|\, H \leq \ker f\}.$$

Show that $H^\sharp$ is a subgroup of $\widehat{G}$ and that if $a \in G$ then

$$\frac{1}{|H^\sharp|}\sum_{f \in H^\sharp} f(a) = \begin{cases} 1 & \text{if } a \in H, \\ 0 & \text{if } a \notin H. \end{cases}$$

7. Determine the number of solutions (mod n) to the congruence $x^2 + 17x + 47 \equiv 0 \pmod{n}$ if: **a.** $n = 3573077$ **b.** $n = 617611061$ **c.** $n = 47394585673492$

8. Determine the set of odd primes p for which $(10|p) = 1$.

9.Let p be an odd prime.

- **a.** Show that -4 is a fourth power mod p if and only if $p \equiv 1 \pmod 4$. [*Suggestion:* Factor the polynomial $x^4 + 4$.]
- **b.** Show that -1 is a fourth power mod p if and only if $p \equiv 1 \pmod 8$.

10. A natural number n is called a *palindrome* if its (base 10) digits read the same forward and backward. So, for example, 10301 is a palindrome, while 12342 is not. Let $P(x)$ denote the number of palindromes less than or equal to x.

- **a.** Show that for $N \in \mathbb{N}$, there is a bijection between the palindromes in $[10^{N-1}, 10^N)$ and the natural numbers in $[10^{M-1}, 10^M)$, where $M = \lfloor (N+1)/2 \rfloor$. [*Suggestion:* Consider the N even and N odd cases separately.]
- **b.** With M and N as above, show that $P(10^N - 1) - P(10^{N-1} - 1) = 9 \cdot 10^{M-1}$.
- **c.** Use part **b** to show that $P(10^N) = O(10^{N/2})$. [*Suggestion:* Use the fact that $P(10^N) = P(10^N - 1)$ and write the latter as a telescoping sum.]
- **d.** Conclude that $P(n) = O(\sqrt{n})$.