Number Theory II — Assignment 4.2
Fall 2010 — Due September 22

Exercise 1. Interpret, and then verify, the following equalities. Throughout, $g(x)$ and $h(x)$ denote nonnegative functions.

a. $O(g(x)) + O(h(x)) = O(\max\{g(x), h(x)\})$.
b. $g(x)O(h(x)) = O(g(x)h(x)) = O(g(x))\,O(h(x))$.
c. $O(O(g(x))) = O(g(x))$.

Exercise 2.

a. Let $f(x)$ and $g(x)$ be polynomials with real coefficients of degrees m and n, respectively. Prove that

$$\lim_{x\to\infty} \frac{f(x)x^{n-m}}{g(x)}$$

exists and is nonzero.

b. Prove that

$$\frac{f(x)}{g(x)} = O\left(x^{m-n}\right)$$

for all sufficiently large x. What does the implied constant depend on?

Exercise 3. Let $\epsilon > 0$. Prove that

$$\log x = O(x^{\epsilon})$$

for all sufficiently large x. Does the implied constant depend on ϵ?