Number Theory II
Fall 2010

Assignment 6.2
Due October 6

Exercise 1. Suppose that f satisfies all of the hypotheses of the Euler Summation Formula, and that in addition

(i) $\lim_{x \to \infty} f(x) = 0$;

(ii) there is an M so that f is monotone for $x \geq M$.

Prove that

$$\sum_{1 < n \leq x} f(n) = \int_1^x f(t)\,dt + A(f) + O\left(|f(x)|\right)$$

where $A(f)$ is a constant that depends only on f.

Exercise 2. Given $n \in \mathbb{N}$, let $\mathcal{F}_n$ denote the set of rational numbers in $[0,1]$ that can be written in the form a/b with $1 \leq b \leq n$. $\mathcal{F}_n$ is known as the set of *Farey fractions of order n*. So, for example

$$\begin{aligned}
\mathcal{F}_1 &= \{0, 1\} \\
\mathcal{F}_2 &= \left\{0, \frac{1}{2}, 1\right\} \\
\mathcal{F}_3 &= \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\} \\
\mathcal{F}_4 &= \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\}.
\end{aligned}$$

Find an asymptotic expression for $|\mathcal{F}_n|$. [*Suggestion:* Relate $|\mathcal{F}_n|$ to $\overline{\varphi}(n)$.]